3.2. Governing Equation and Boundary Conditions

3.2.1. governing equation

If flow is assumed to be incompressible and irrotational and the fluid is to be inviscid, the flow motion is governed by the Laplace equation. That is,

$$\nabla^2\Phi = 0, \quad -h < z < \zeta$$

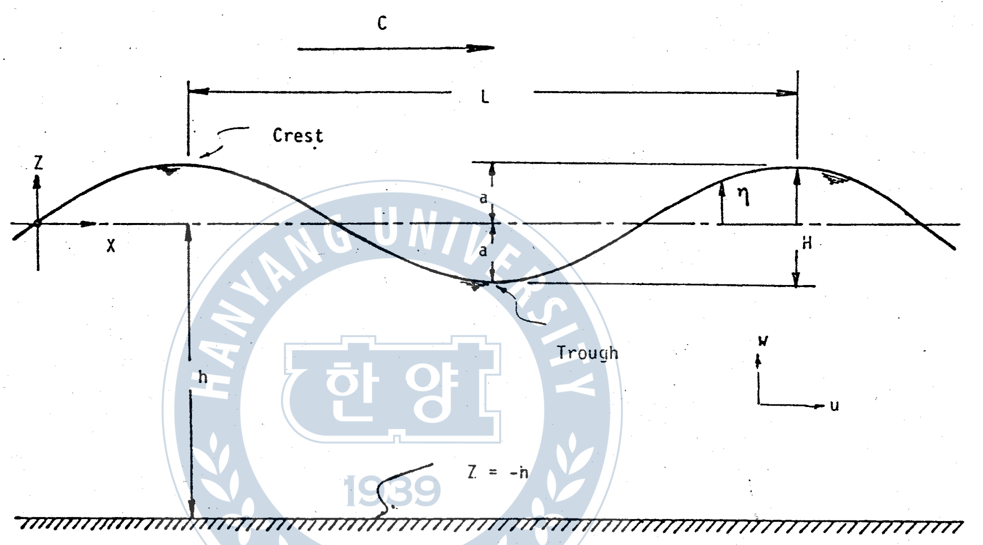

Fig. 2 미소 진폭파의 요소

3.2.2. boundary conditions

a. kinematic free surface boundary condition (KFSBC)

KFSBC physically means that a water particle on the free surface can not jump, so it should remain on the free surface. That is, the water particle cannot penetrate into the free surface. If the free surface is defined as $z = \zeta(x, t)$ or $F(x, z, t) = \zeta(x, t) - z = 0$, we then have

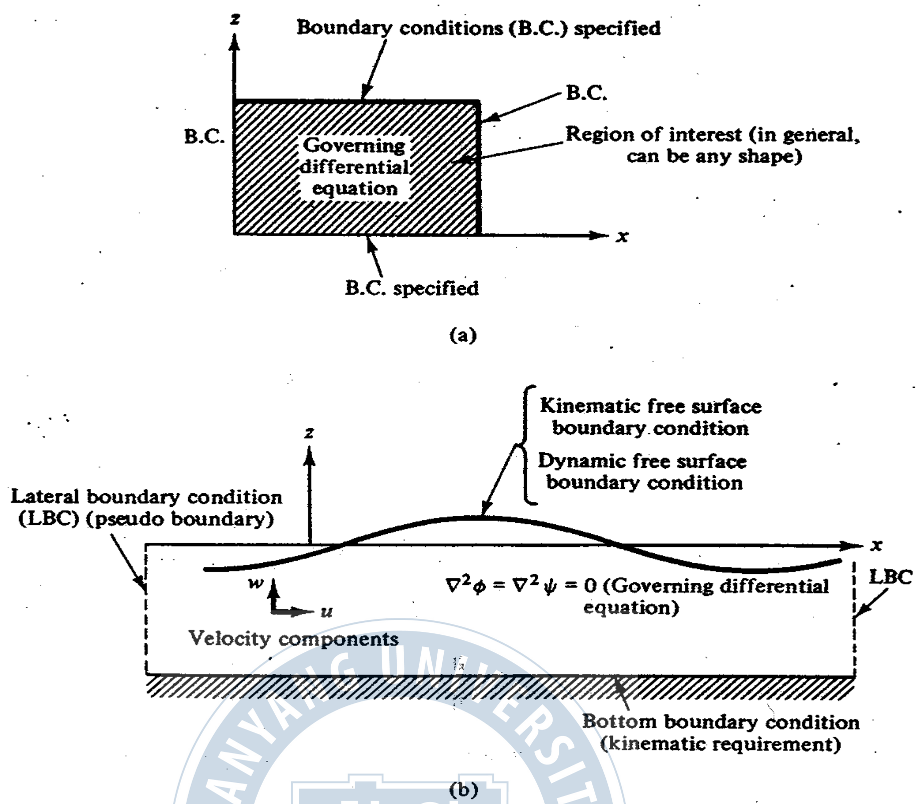

Figure 3.1 (a) General structure of two-dimensional boundary value problems. (*Note*: The number of boundary conditions required depends on the order of the differential equation.) (b) Two-dimensional water waves specified as a boundary value problem.

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathrm{u}\cdot\nabla \mathrm{F}$$

$$= \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + w\frac{\partial F}{\partial z}$$

$$= \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} - w = 0, \quad z = \zeta(x,t)$$

It should be noted again that ζ is not a function of z.

If the boundary is described by $F(x, y, z, t) = 0$, where we have not only spatial dependency but also time dependency because the location of the boundary is changing in space and time, the rate of change of F for an observer moving with the boundary is zero. Using the chain rule of differentiation , we have

$$\frac{DF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

in which $(dx/dt, dy/dt, dz/dt) = \mathbf{u}$ is the velocity of the boundary. Then, the above equation can be rewritten as

$$\frac{DF}{dt} = \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F = 0, \quad \text{on} \quad F = 0$$

b. dynamic free surface boundary condition (DFSBC)

DFSBC is simply the Bernoulli equation on the free surface. Therefore, DFSBC is expressed as

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla \Phi|^2 + \frac{p_a}{\rho} + gz = 0, \quad z = \zeta(x, y, t)$$

in which p_a is the atmospheric pressure on the free surface and can be ignored. Then, we have

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla \Phi|^2 + gz = 0, \quad z = \zeta$$

c. solid boundary condition (SBC)

SBC is a boundary condition applied on the solid surface. In general, SBC can be derived in a similar fashion to KFSBC. Thus, we have SBC over a spatially varying depth as

$$u\frac{\partial h}{\partial x} + w = 0, \quad z = -h(x)$$

If the bottom topography is constant, that is h = constant, SBC can be simplified to be

$$w = 0, \quad z = -h$$

SBC also means that the water particle cannot penetrate into the solid boundary and SBC is also called BBC (bottom boundary condition) if it is applied to the bottom boundary.

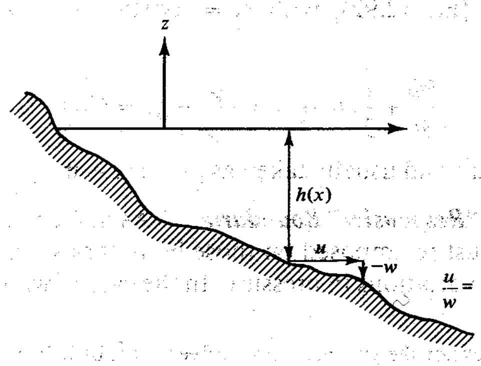

Figure 3.3 Illustration of bottom boundary condition for the two-dimensional case.

Figure 3.7 Boundary value problem specification for periodic water waves.

3.2.3. linearization of boundary conditions

a. nonlinearity in two aspects

- The boundary conditions contain nonlinear terms such as $u\partial_x\zeta$ and $|\nabla\Phi|^2$.
- The free surface boundary conditions are applied on an unknown location in space. The position of the free surface is actually a part of solution.

b. The main objective of linearization is, of course, to simplify the problem.

c. The basic assumption of the small amplitude wave theory is that the wave motion is small and then water particle velocity and the free surface displacement are also small.

d. Moreover, since the free surface does not deviate much from its rest position, we can apply the free surface boundary conditions (KFSBC and DFSBC) on the still water level $(z=0)$

instead of on the free surface ($z=\zeta$).

e. After neglecting the nonlinear terms the boundary conditions can be summarized as:

- KFSBC

$$\frac{\partial \zeta}{\partial t} - w = 0, \quad z = 0$$

- DFSBC

$$\frac{\partial \Phi}{\partial t} + g\zeta = 0, \quad z = 0$$

- SBC

$$w = 0, \quad z = -h$$

f. Since the nonlinear terms are all neglected, the small amplitude wave theory becomes linear. It must be noted that the domain of governing equation also changes from $-h < z < \zeta$ to $-h < z < 0$.

g. The nonlinear terms can also ignored by a more formal way. To show this we first introduce three important variables: ω, L and a. Then, the typical time and the length scales may be ω^{-1} and L, respectively. Of course, a is another length scale representing the amplitude of a wave. We also introduce a symbol for the order of magnitude, O. Then, $\zeta \sim O(a)$ means that ζ has the order of magnitude of a.

h. Since we already have both typical time and length scales, we can normalize the governing equation and boundary conditions using the following nondimensional variables:

$$t' = \omega t, \quad (x', z') = k(x, z), \quad \zeta' = \frac{1}{a}\zeta, \quad \Phi' = \frac{k}{a\omega}\Phi$$

in which k is the wavenumber defined as $k = 2\pi / L$.

i. Normalized (or dimensionless) governing equation and boundary conditions are:

- governing equation

$$\nabla'^2 \Phi' = 0, \quad -h' < z' < (ka)\zeta'$$

- KFSBC

$$\frac{\partial \zeta'}{\partial t'} + (ka)u' \frac{\partial \zeta'}{\partial x'} - w' = 0, \quad z' = (ka)\zeta'$$

- DFSBC

$$\frac{\partial \Phi'}{\partial t'} + \frac{1}{2}(ka)(u'^2 + w'^2) + \frac{kg}{\omega^2}\zeta' = 0, \quad z' = (ka)\zeta'$$

- SBC

$$w' = 0, \quad z' = -h'$$

j. In the small amplitude wave theory, the order of magnitude of wave steepness, $O(ka)$ (actually, wave steepness is defined as H/L) is assumed to be very small. In other words, the wave amplitude (a) is much smaller than the wavelength (L). The order of magnitude of $O(kg/\omega^2)$ is yet determined so it should be included. After neglecting terms of $O(ka)$ the governing equation and boundary conditions are finally summarized as:

- governing equation

$$\nabla^2 \Phi = 0, \quad -h < z < 0$$

- KFSBC

$$\frac{\partial \zeta}{\partial t} - w = 0, \quad z = 0$$

- DFSBC

$$\frac{\partial \Phi}{\partial t} + g\zeta = 0, \quad z = 0$$

- SBC

$$w = 0, \ \ z = -h$$

in which all dimensions are recovered. In some texts, a terminology of CFSBC (Combined Free Surface Boundary conditions) is also used. CFSBC is a combined form of KFSBC and DFSBC. That is, by deleting ζ or Φ, KFSBC and DFSBC can be combined as:

$$\frac{\partial^2 \Phi}{\partial t^2} + gw = 0, \ \ z = 0$$