# Discrete Mathematics: Relations

# representing relations

- some special ways to represent binary relations:
  – with a zero-one matrix
  – with a directed graph.

# zero-one matrices

- a binary relation $R: A \times B$ can be represented by a matrix $M_R = [m_{ij}]$

  $m_{ij} = 1$ if $(a_i, b_j) \in R$

  $m_{ij} = 0$ if $(a_i, b_j) \notin R$

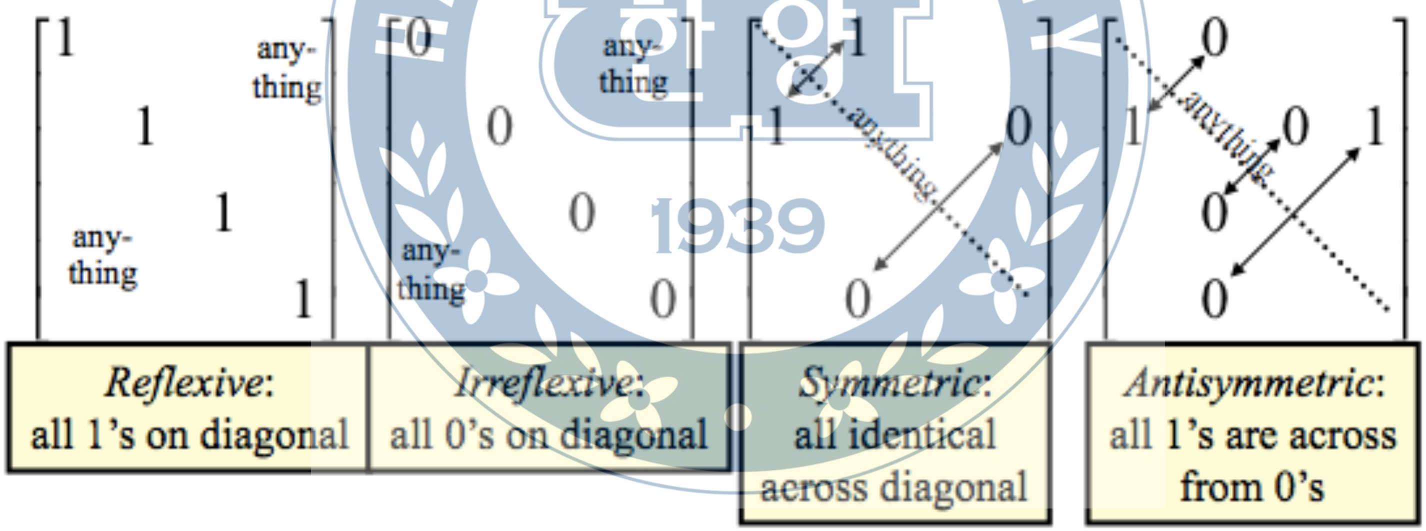
A = {Joe, Fred, Mark}, B = {Susan, Mary, Sally}

suppose Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally

| | Susan | Mary | Sally |
|---|---|---|---|
| Joe | 1 | 1 | 0 |
| Fred | 0 | 1 | 0 |
| Mark | 0 | 0 | 1 |

# zero-one matrices

- properties of the relation, reflexive, irreflexive, symmetric, and antisymmetric, are very easy to recognize by inspection of the zero-one matrix.



# zero-one matrices

the relation R on a set is represented by the zero-one matrix

$$M_R = \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}$$

reflexive?
symmetric?
antisymmetric?

# combining relations

- combining relations by set operations
- set operations: union, intersection, and difference

$A = \{1, 2, 3\}$ and $B = \{u, v\}$

$R1 = \{(1, u), (2, u), (2, v), (3, u)\}$

$R2 = \{(1, v), (3, u), (3, v)\}$

$R1 \cup R2 = \{(1, u), (2, u), (2, v), (3, u), (1, v), (3, v)\}$

$R1 \cap R2 = \{(3, u)\}$

$R1 - R2 = \{(1, u), (2, u), (2, v)\}$

$R2 - R1 = \{(1, v), (3, v)\}$

# combining relations

- union of two relations R1 and R2 can be represented in terms of matrix operations

$m_{ij} = a_{ij} \vee b_{ij}$ for all i and j

A = { 1, 2, 3} and B = {u, v}

R1 = {(1, u), (2, u), (2, v), (3, u)}

R2 = {(1, v), (3, u), (3, v)}

$$M_{R1} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{vmatrix} \qquad M_{R2} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{vmatrix} \qquad M_{R1 \cup R2} = M_{R1} \vee M_{R2} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{vmatrix}$$

# combining relations

- intersection of two relations R1 and R2 can be represented in terms of matrix operations

$m_{ij} = a_{ij} \wedge b_{ij}$ for all i and j

A = { 1, 2, 3} and B = {u, v}

R1 = {(1, u), (2, u), (2, v), (3, u)}

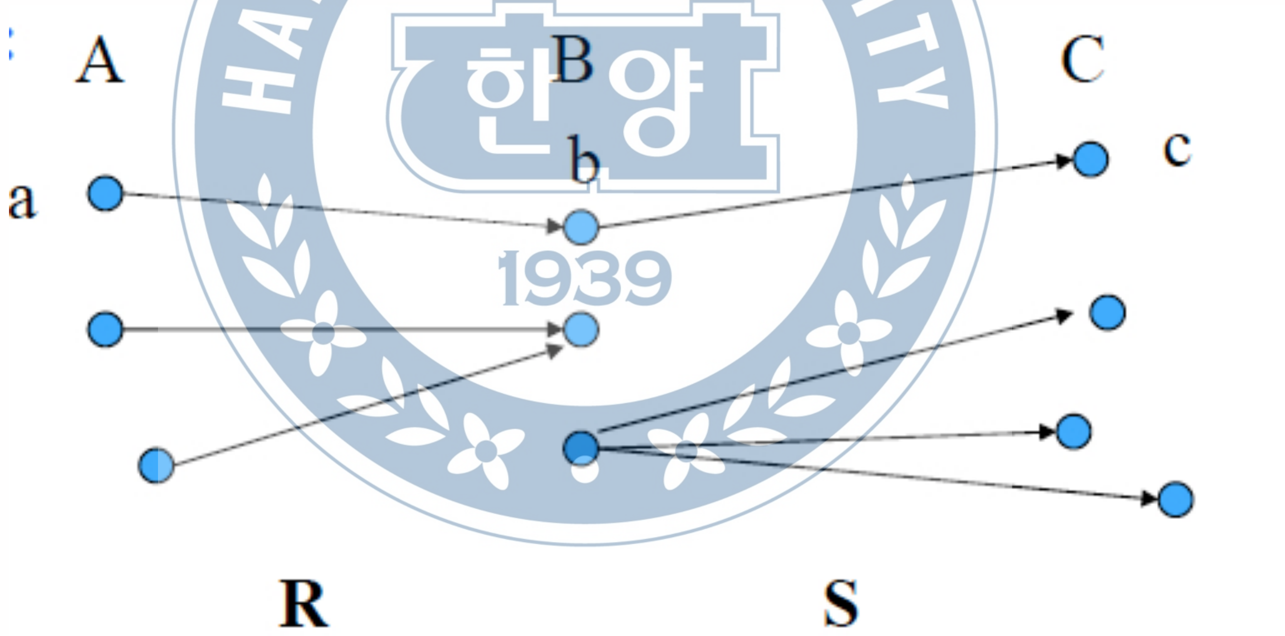
R2 = {(1, v), (3, u), (3, v)}

$$M_{R1} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{vmatrix} \qquad M_{R2} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{vmatrix} \qquad M_{R1 \cap R2} = M_{R1} \wedge M_{R2} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{vmatrix}$$

# combining relations

- let R: A × B, and S: B × C. Then the *composite* S ○ R of R and S is defined as:

  S ○ R = {(a,c) | ∃b: aRb ∧ bSc}



# combining relations

boolean product of two relations R1(m x n) and R2(n x p) can be represented in terms of matrix operations

$m_{ij} = 1$ if $a_{ij} = 1$ and $b_{jk} = 1$ for $k = 1, 2, \ldots, n$

$0$ otherwise

A = {1, 2}, B = {1, 2, 3} C = {a, b}

R (a relation from A to B) = {(1, 2), (1, 3), (2, 1)}

S (a relation from B to C) = {(1, a), (3, b), (3, a)}

S ○ R = {(1, a), (1, b), (2, a)}

$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$ $M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$ $M_{S \circ R}$ $M_R \odot M_S = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$

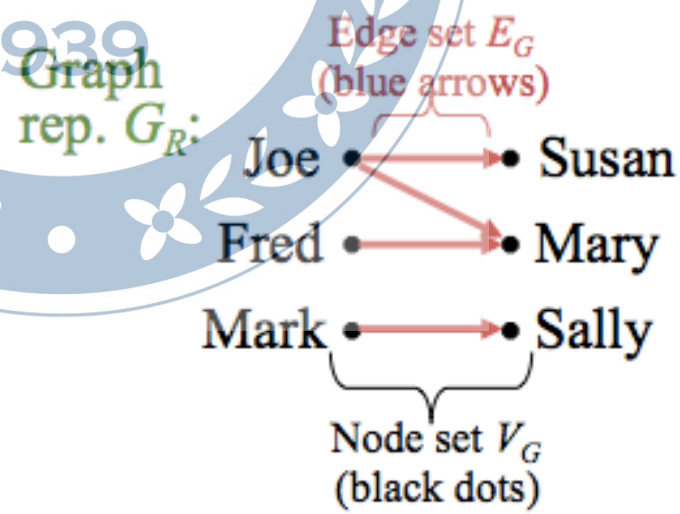
# directed graphs

- a directed graph or digraph $G=(V_G,E_G)$ is a set $V_G$ of vertices (nodes) with a set $E_G \subseteq V_G \times V_G$ of edges (*arcs*, *links*).
- visually represented using dots for nodes, and arrows for edges
- notice that a relation $R:A\times B$ can be represented as a graph $G_R=(V_G=A\cup B, E_G=R)$.



Matrix representation $M_R$:

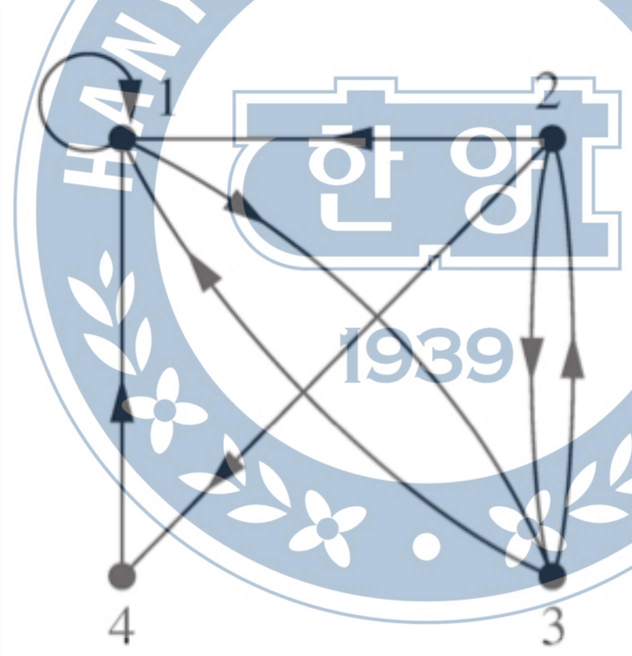
| | Susan | Mary | Sally |
|---|---|---|---|
| Joe | 1 | 1 | 0 |
| Fred | 0 | 1 | 0 |
| Mark | 0 | 0 | 1 |



# directed graphs

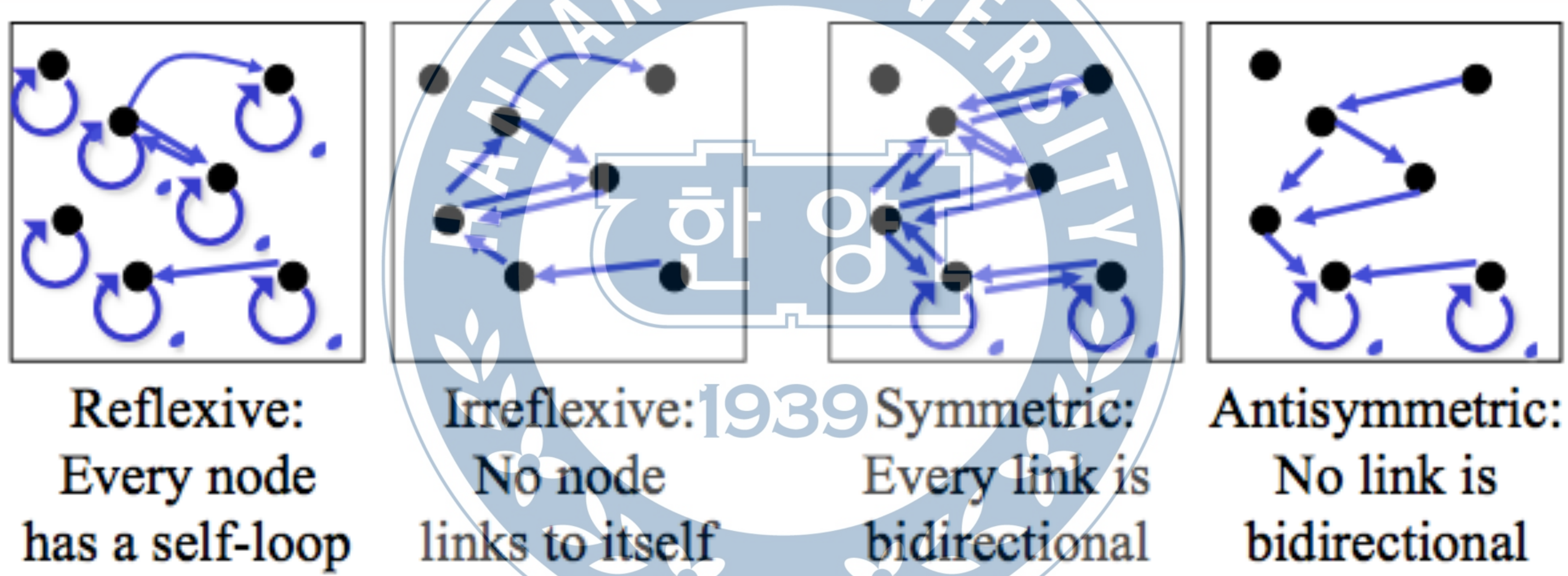
R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)}

on the set {1, 2, 3, 4}



# digraph reflexive, symmetric

- it is easy to recognize the reflexive, irreflexive, symmetric and antisymmetric properties by graph inspection.



# closures of relations

- for any property X, the "X closure" of a set A is defined as the "smallest" superset of A that has the given property.
- reflexive closure
- symmetric closure
- transitive closure

# reflexive closure

- the reflexive closure of a relation R on A is obtained by adding (a,a) to R for each a∈A; i.e., it is $R \cup I_A$

R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3}

reflexive closure of R = {(1, 1), (1, 2), (2, 1), (3, 2), (2, 2), (3, 3)}

R = {(a, b) | a < b} on the set of integers is not reflexive

reflexive closure of R = {(a, b) | a ≤ b}

# symmetric closure

- the symmetric closure of R is obtained by adding (b,a) to R for each (a,b) in R; i.e., it is $R \cup R^{-1}$

R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3}

symmetric closure of R = {(1, 1), (1, 2), (2, 1), (3, 2), (2, 3)}

R = {(a, b) | a < b} on the set of integers is not symmetric

symmetric closure of R is $R \cup R^{-1}$= {(a, b) | a > b} ∪ {(b, a) | a > b}

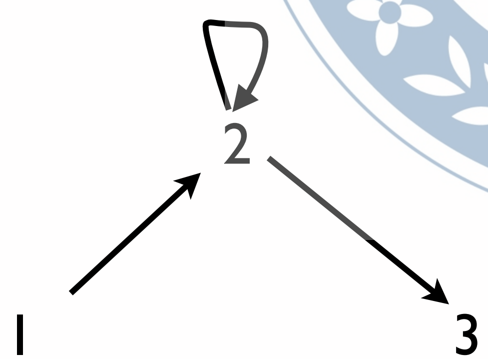
= {(a, b) | a≠ b}

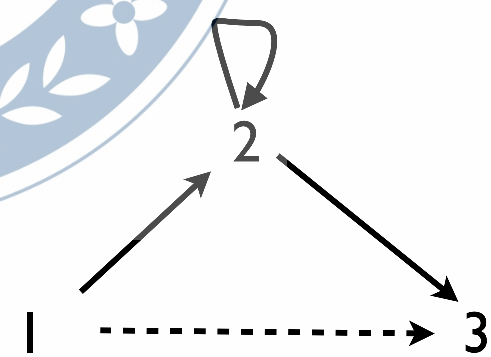
# transitive closures

- the transitive closure or connectivity relation of R is obtained by repeatedly adding (a,c) to R for each (a,b),(b,c) in R.
- Finding a transitive closure is to find all pairs of elements that are connected with a directed path

R = {(1, 2), (2, 2), (2, 3)} on the set A = {1, 2, 3} is not transitive.

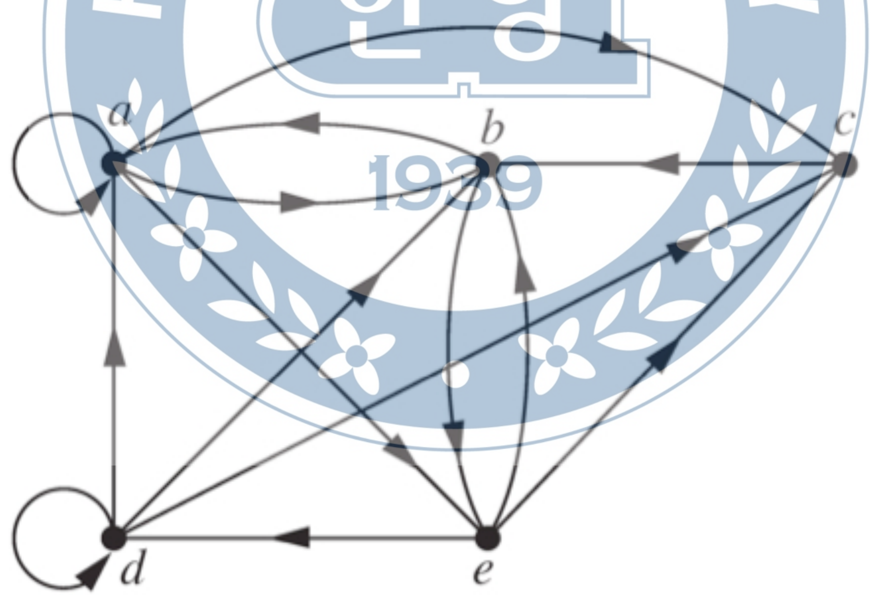
transitive closure of R = {(1, 2), (2, 2), (2, 3), (1, 3)}



# paths in digraphs

- a path of length n from node a to b in the directed graph G is a sequence $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$ of n ordered pairs in $E_G$
- a path of length $n \geq 1$ from a to itself is called a circuit or a cycle.



# paths in digraphs

- note that there exists a path of length n from a to b in R if and only if $(a,b) \in R^n$.

length=1: there is a path from a to b of length 1 if and only if $(a, b) \in R$

length=n: assume that the theorem is true

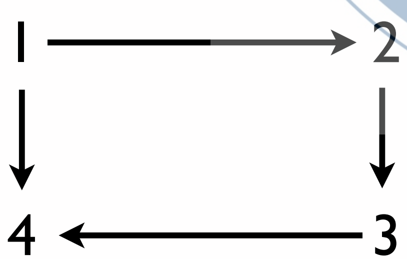
length=n+1: $c \in A$ such that there is a path of length one from a to c, $(a, c) \in R$ and a path of length n from c to b, $(c, b) \in R^n$. By the inductive hypothesis, there is a path of length n+1 from a to b iff there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. Therefore, there is a path of length n+1 form a to b iff $(a, b) \in R^{n+1}$

# connectivity relation

- Let R be a relation on a set A. The connectivity relation $R^*$ consists of all pairs (a, b) such that there is a path of any length between a and b in R

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

R = {(1, 2), (2, 3), (3, 4), (1, 4)} on the set A={1, 2, 3, 4}




$R^2$ = {(1, 3), (2, 4)}

$R^3$ = {(1, 4)}

$R^4$ = ?

:

$R^*$ = ?

# connectivity relation

- Let R be a relation on a set A. The connectivity relation $R^*$ consists of all pairs (a, b) such that there is a path of any length between a and b in R

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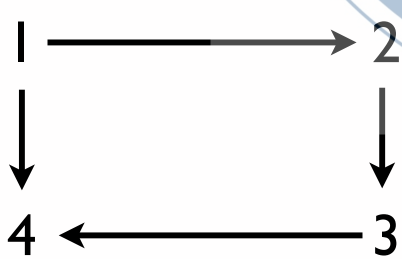


$R^2$ = {(1, 3), (2, 4)}

$R^3$ = {(1, 4)}

$R^4$ = {}

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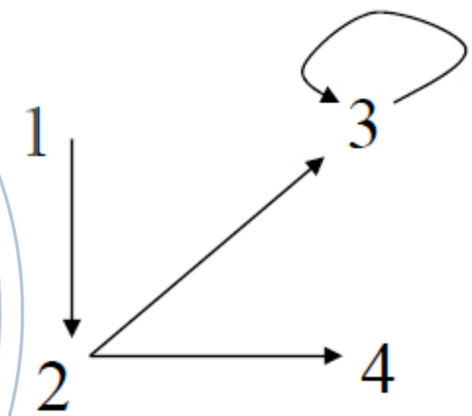
$R^*$ = {(1, 2), (2, 3), (3, 4), (1, 4),

# powers of R

R = {(1, 2), (2, 3), (2, 4), (3, 3)} on the set A={1, 2, 3, 4}




R = {(1, 2), (2, 3), (2, 4), (3, 3)}

$R^2$ = {(1, 3), (1, 4), (2, 3), (3, 3)}

$R^3$ = {(1, 3), (2, 3), (3, 3)}

$R^4$ = {(1, 3), (2, 3), (3, 3)}

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# transitive closures

- $M_R$ is the zero-one matrix of the relation R on a set with n elements. the zero-one matrix of the transitive closure R* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

Find the zero-one matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

# transitive closures

$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$

procedure transitive closure ($M_R$: zero-one n x n matrix)
A := $M_R$
B := A
for i := 2 to n
    A := A ⊙ $M_R$    $n^2(2n-1)(n-1)$
    B := B ∨ A    $n^2(n-1)$
return B {B is the zero-one matrix for R*}

$n^2(2n-1)(n-1)$ bit operations

- $n^2$: for each $M_R$
- $(2n-1)$: for multiplication and addition for $i^{th}$ row and $j^{th}$ column
- $(n-1)$: for loop