# Chapter 3. Euclidean vector space

Vectors in 2-dimensional and 3-dimensional spaces (*i.e.*, $xy$-plane and $xyz$-space, respectively) could be understood either as vectors in geometry or as vectors in a coordinate system. Since we are familiar to the 2-D and 3-D spaces through the geometry, vectors in 2-D and 3-D spaces could have geometrical meaning and thus make the understanding of vectors much easier.

## 3.1 Vectors in 2-D and 3-D spaces

### 3.1-1 Geometrical representation of vectors

A *vector* is an entity which has both magnitude and direction. In a geometrical approach, it is represented by a directed line segment or arrow that corresponds to a displacement from one point *A* (initial point) to another point *B* (terminal point). We denote it by $\boldsymbol{v} = \overrightarrow{\mathrm{AB}}$. In this geometric representation, we can define algebraic operations on vectors as following:

(1) **equal vectors**: vectors with the same length and same direction are equal even though they may be located in some different position (see Figure 3.1).

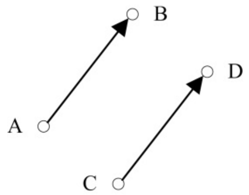


Figure 3.1 Two equal vectors: $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{CD}}$

(2) **vector addition**: If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vectors, the sum $\boldsymbol{v}+\boldsymbol{w}$ is the vector determined as follows: Position the vector $\boldsymbol{w}$ so that its initial point coincides with the terminal point of $\boldsymbol{v}$. The vector $\boldsymbol{v}+\boldsymbol{w}$ is represented by the arrow from the initial point of $\boldsymbol{v}$ to the terminal point of $\boldsymbol{w}$, as shown in Figure 3.2. It is evident that $\boldsymbol{v}+\boldsymbol{w}=\boldsymbol{w}+\boldsymbol{v}$ and the sum coincides with the diagonal of the parallelogram determined by $\boldsymbol{v}$ and $\boldsymbol{w}$ when these vectors are positioned so that they have the same initial point (*parallelogram law*).

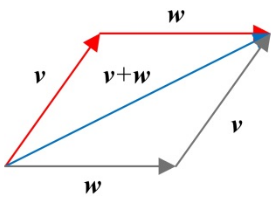


Figure 3.2 Vector addition: $\boldsymbol{v}+\boldsymbol{w}=\boldsymbol{w}+\boldsymbol{v}$

(3) **negative vector**: If $\boldsymbol{v}$ is any non-zero vector, then $-\boldsymbol{v}$, the *negative* of $\boldsymbol{v}$, is the vector that has the same length as $\boldsymbol{v}$ but is heading opposite direction. Then,

$$\boldsymbol{v}+(-\boldsymbol{v})=\boldsymbol{0}$$

where $\boldsymbol{0}$ is the zero vector.

(4) If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vectors, the difference $\boldsymbol{v}-\boldsymbol{w}$ is defined by

$\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-\boldsymbol{w})$

Figure 3.3 illustrates two equivalent representation of the difference of two vectors.

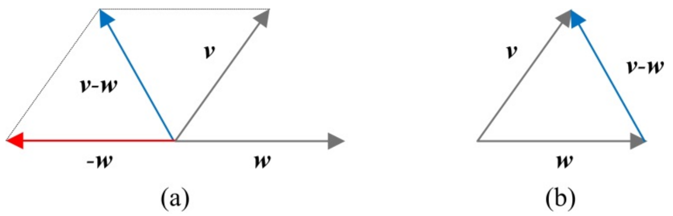


Figure 3.3 Two equivalent representation of vector difference: $\boldsymbol{v} - \boldsymbol{w}$

(4) **scalar multiple**: If $\boldsymbol{v}$ is a non-zero vector and $k$ is non-zero real number (*scalar*), then the product $k\boldsymbol{v}$ (usually called *scalar multiple*) is a vector whose length is $|k|$ times the length of $\boldsymbol{v}$ and whose direction is the same as that of $\boldsymbol{v}$ if $k > 0$ and opposite if $k < 0$. When $k = 0$, $k\boldsymbol{v} = \boldsymbol{0}$.

## 3.1-2 Vectors in coordinate system

A geometrical representation often causes ambiguity since it does not precisely provide information about its characteristic. In particular, there is no restriction on their initial points, which make arithmetic with vectors difficult. This problem can often be simplified by using a coordinate system. In this approach, we restrict the initial point of a vector be at the origin and represent the vector using the coordinates of the terminal point of the vector, so that we can denote a vector with an ordered pair of real numbers; *e.g.* the vector in Figure 3.4 can be represented as

$$\boldsymbol{a} = \overrightarrow{\mathrm{OA}} = (2, 1) \quad \text{or} \quad \boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

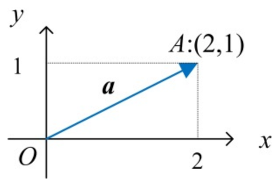


Figure 3.4 A vector in a (rectangular) coordinate system

This vector is assumed to be a 2-dimensional vector. In general, we define $\mathbb{R}^n$ (so called *n*-dimensional *Euclidean vector space*: here $\mathbb{R}$ represents a set of real numbers) as the set of all ordered $n$-tuples of real numbers, written as

$$\boldsymbol{b} = (b_1, b_2, \dots, b_n) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

In this representation, we can define algebraic operations on vectors in terms of vector components.

(1) **equal vectors**: two vectors, $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ are equal, if their corresponding components are the same: *i.e.*

(3.1) $\quad v_1 = w_1$ and $v_2 = w_2$

(2) **vector addition**: If $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$, then

(3.2) $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2)$

(3) **scalar multiple**: If $\boldsymbol{v} = (v_1, v_2)$ is a non-zero vector and $k$ is non-zero real number (*scalar*), then

(3.3) $k\boldsymbol{v} = (kv_1, kv_2)$

(4) **vector difference**:

(3.4) $\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-1)\boldsymbol{w} = (v_1 - w_1, v_2 - w_2)$

As we can see, operations on vectors are defined as component-wise operations. In vector arithmetic, the vector addition and the scalar multiplication are two main operations. Their combined operations is called the *linear combination* of vectors; *i.e.*

(3.5) $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots \;\; c_k\boldsymbol{v}_k$

where the scalars, $c_1, c_2, \ldots, c_k$, are called the *coefficients* of the linear combination. Also, these two operations obey the general arithmetic rule (in component-wise) defined on the set of real numbers.

***Theorem 3.1*** (*Algebraic properties of vectors in*$\mathbb{R}^n$) Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in $\mathbb{R}^n$. Then, for scalars $c$ and $d$,

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ (*commutative rule*)
2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$ (*associative rule*)
3. $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$ (additive identity)
4. $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$ (*additive inverse*)
5. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ (*distributive rule*)
6. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$ (*distributive rule*)
7. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
8. $1 \cdot \boldsymbol{u} = \boldsymbol{u}$ (multiplicative identity)

## 3.2 Norm, dot product, and distance in $\mathbb{R}^n$

The term, Euclidean vector space $\mathbb{R}^n$, refers to an $n$-dimensional vector space where we can relate some geometrical concepts to vectors. Recall that a vector is an entity with length and direction. We want to express these geometrical concepts in terms of mathematical notations.

### 3.2-1 Dot product

At first, we start from the length of a vector.

***Definition*** The *dot product* $\boldsymbol{u} \cdot \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ is defined by

(3.6) $\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + \cdots u_nv_n$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$.

If we denote each vector as column vector as

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then we the dot product become

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$$

Later, in a general vector space, we will denote it as the *inner product*. Therefore, the dot product is an inner product defined on $\mathbb{R}^n$. The dot product function has the following arithmetic properties.

**Example 3.1** When $\boldsymbol{u} = (1, 2, -3)$ and $\boldsymbol{v} = (-3, 5, 2)$, $\boldsymbol{u} \cdot \boldsymbol{v} = 1 \cdot (-3) + 2 \cdot 5 + (-3) \cdot 2 = 1$

***Theorem 3.2*** Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in $\mathbb{R}^n$ and $c$ be a scalar. Then;

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ (*commutative rule*)
2. $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ (*distributive rule*)
3. $(c\boldsymbol{u}) \cdot \boldsymbol{v} = c(\boldsymbol{u} \cdot \boldsymbol{v})$
4. $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$, and $\boldsymbol{u} \cdot \boldsymbol{u} = 0$, if and only if $\boldsymbol{u} = \boldsymbol{0}$.

## 3.2-2 Length

In 2-dimensional Euclidean vector space $\mathbb{R}^2$, the *length* of a vector $\boldsymbol{v} = (v_1, v_2)$ is the distance from the origin to the terminal point, which is

$$\sqrt{v_1^2 + v_2^2} = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$$

The term, *norm*, is a mathematical generalization of the geometrical quantity, length, so that we can extend this concept beyond $\mathbb{R}^3$.

***Definition*** The *norm* of a vector, $\boldsymbol{v} = (v_1, v_2, \dots, v_n)$ in $\mathbb{R}^n$, is the non-negative scalar $\|\boldsymbol{v}\|$ defined by

(3.7) $$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

***Theorem 3.3*** Let $\boldsymbol{v}$ be a vector in $\mathbb{R}^n$ and $c$ be a scalar. Then,

1. $\|\boldsymbol{v}\| = 0$, if and only if $\boldsymbol{v} = \boldsymbol{0}$.
2. $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$

(1) A vector of length-1 is called a *unit vector*.

(2) Given any non-zero vector $\boldsymbol{v}$, we can always find a unit vector in the same direction as $\boldsymbol{v}$ by dividing $\boldsymbol{v}$ by its own length. This is referred as *normalizing* a vector.

(3) In $\mathbb{R}^n$, the *standard unit vector* is a set of vectors, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$, where $\boldsymbol{e}_i$ is an all-zero vector except 1 in its $i^{th}$ component.

***Theorem 3.4*** (*Cauchy-Schwarz inequality*) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{R}^n$. Then

(3.8) $$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

(Proof) Consider, for a scalar $t$, $\|t\boldsymbol{u} + \boldsymbol{v}\|^2 \geq 0$, for all $t$.

Since $\|t\boldsymbol{u} + \boldsymbol{v}\|^2 = (t\boldsymbol{u} + \boldsymbol{v}) \cdot (t\boldsymbol{u} + \boldsymbol{v}) = t^2\|\boldsymbol{u}\|^2 + 2t\boldsymbol{u} \cdot \boldsymbol{v} + \|\boldsymbol{v}\|^2$ is a quadratic function of $t$ and is non-negative for all $t$, we must have

$$|\boldsymbol{u}\cdot\boldsymbol{v}|^2 \le \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \quad \blacksquare$$

When $\boldsymbol{u}=(u_1,u_2)$ and $\boldsymbol{v}=(v_1,v_2)$, (3.8) implies

$$|u_1v_1+u_2v_2|^2 \le (u_1^2+u_2^2)(v_1^2+v_2^2)$$

Note that the equality holds when $\boldsymbol{u}=k\boldsymbol{v}$.

***Theorem 3.5*** (*Triangle inequality*) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{R}^n$. Then

(3.9) $\quad \|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$

(Proof) Since $\|\boldsymbol{u}+\boldsymbol{v}\|^2=(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{v}=\|\boldsymbol{u}\|^2+2\boldsymbol{u}\cdot\boldsymbol{v}+\|\boldsymbol{v}\|^2$

by the Cauchy-Schwarz inequality, we have

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 \le \|\boldsymbol{u}\|^2+2|\boldsymbol{u}\cdot\boldsymbol{v}|+\|\boldsymbol{v}\|^2 \le \|\boldsymbol{u}\|^2+2\|\boldsymbol{u}\|\|\boldsymbol{v}\|+\|\boldsymbol{v}\|^2=(\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2$$

In geometry, we can construct the vector $\boldsymbol{u}+\boldsymbol{v}$ by connecting the initial point of the vector $\boldsymbol{u}$ with the terminal point of the vector $\boldsymbol{v}$ while we place the initial point of $\boldsymbol{v}$ at the terminal point of $\boldsymbol{u}$ as depicted in Figure 3.5. Then, norm of each vectors are the length of them.

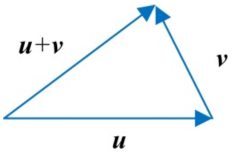


Figure 3.5 Geometrical illustration of triangular inequality

### 3.2-3 Distance

In 2-dimensional Euclidean vector space $\mathbb{R}^2$, the *distance* between two vectors $\boldsymbol{u}=(u_1,u_2)$ and $\boldsymbol{v}=(v_1,v_2)$, denoted by $d(\boldsymbol{u},\boldsymbol{v})$, is the distance from terminal points of each vectors: *i.e.*

$$d(\boldsymbol{u},\boldsymbol{v})=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$$

As we can see in Figure 3.3(b), it is the length of the difference vector, $\boldsymbol{u}-\boldsymbol{v}$. Thus, we can express the distance in terms of norm.

***Definition*** The distance (or *metric*), $d(\boldsymbol{u},\boldsymbol{v})$, between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ is defined by

(3.10) $\quad d(\boldsymbol{u},\boldsymbol{v})=\|\boldsymbol{u}-\boldsymbol{v}\|$

**Example 3.2** When $\boldsymbol{u}=(1,2,-3)$ and $\boldsymbol{v}=(-3,5,2)$, the distance between two vectors is

$$d(\boldsymbol{u},\boldsymbol{v})=\|1-(-3),2-5,(-3)-2\|=\sqrt{4^2+(-3)^2+(-5)^2}=5\sqrt{2}$$

### 3.2-4 Angle

Consider two vectors $\boldsymbol{u}=(u_1,u_2)$ and $\boldsymbol{v}=(v_1,v_2)$ in a 2-dimensional Euclidean vector space $\mathbb{R}^2$, as seen in Figure 3.6. By applying the law of cosines to a triangle constructed by three vectors, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}-\boldsymbol{v}$, we have

$$c^2=(b-a\cdot cos\theta)^2+(a\cdot sin\theta)^2=b^2+a^2-2ba\cdot cos\theta$$

or

$$\|\boldsymbol{u}-\boldsymbol{v}\|^2=\|\boldsymbol{v}\|^2+\|\boldsymbol{u}\|^2-2\|\boldsymbol{v}\|\|\boldsymbol{u}\|cos\theta$$

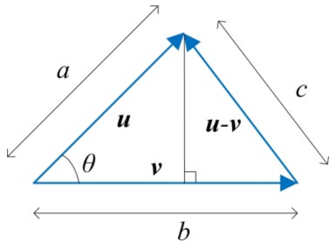


Figure 3.6 Geometrical illustration of angle between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:
$a = \|\boldsymbol{u}\|,\ b = \|\boldsymbol{v}\|,\ c = \|\boldsymbol{u} - \boldsymbol{v}\|$

so that the angle between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are given by

(3.11) $$cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

**Example 3.3** When $\boldsymbol{u} = (2, 1, -2)$ and $\boldsymbol{v} = (1, 1, 1)$, the angle between two vectors is

$$cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{2\cdot1+1\cdot1+(-2)\cdot1}{\sqrt{2^2+1^2+(-2)^2}\sqrt{1^2+1^2+1^2}} = \frac{1}{3\sqrt{3}}$$

In geometry, we say two vectors are perpendicular or in right angle, if the angle between two vectors are is 90°. If $\theta = 90°$ or $270°$, $cos\theta = 0$ and so $\boldsymbol{u}\cdot\boldsymbol{v} = 0$. In linear algebra, we make use of this relation to define perpendicular vectors.

***Definition*** Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ are *orthogonal* to each other, if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$. We usually denote this relation by $\boldsymbol{u} \perp \boldsymbol{v}$. We agree that the zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$. A non-empty set of vectors in $\mathbb{R}^n$ is called an *orthogonal set* if all pairs of distinct vectors in the set are orthogonal.

If two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^2$ are *orthogonal* to each other, then they construct a right triangle so that we can define the Pythagorean theorem as

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{u}\|^2, \text{ if and only if } \boldsymbol{u} \perp \boldsymbol{v}.$$

### 3.2-5 Orthogonal Projection

Suppose we want to find the distance between from a point $B$ to a line $\ell$ in $\mathbb{R}^2$. As shown in Figure 3.7, it is the length of the perpendicular line segment $\overline{PB}$. The vector, $\overrightarrow{AP}$, is called the *projection* of $\overrightarrow{AB}$ onto the line $\ell$.

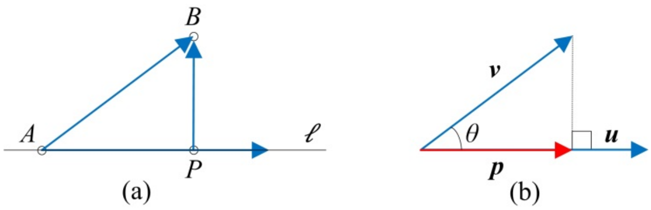


Figure 3.7 Orthogonal projection

Consider two non-zero vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$. Let $\boldsymbol{p}$ be the vector obtained by dropping a perpendicular line from the head of $\boldsymbol{v}$ onto $\boldsymbol{u}$, as in Figure 3.7(b). Then,

$$\boldsymbol{p} = \|\boldsymbol{p}\|\hat{\boldsymbol{u}}, \text{ where } \hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$

and

$$\|\boldsymbol{p}\| = \|\boldsymbol{v}\|\cos\theta, \text{ where } \cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

so that

(3.12) $$\boldsymbol{p} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|^2}\boldsymbol{u} = proj_{\boldsymbol{u}}(\boldsymbol{v})$$

This is the *orthogonal projection* of $\boldsymbol{v}$ onto $\boldsymbol{u}$, and denoted by $proj_{\boldsymbol{u}}(\boldsymbol{v})$.

***Theorem 3.6*** (*Projection theorem*) If $\boldsymbol{u}$ and $\boldsymbol{a}$ are vectors in $\mathbb{R}^n$, and if $\boldsymbol{a} \neq \boldsymbol{0}$, then $\boldsymbol{u}$ can be expressed in exactly one way in the form $\boldsymbol{u} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, where $\boldsymbol{w}_1$ is a scalar multiple of $\boldsymbol{a}$ and $\boldsymbol{w}_2$ is orthogonal to $\boldsymbol{a}$.

In this theorem, each component can be expressed as

$$\boldsymbol{v}_1 = \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = proj_{\boldsymbol{a}}(\boldsymbol{u}) \text{ (orthogonal projection of } \boldsymbol{u} \text{ onto } \boldsymbol{a})$$

and

$$\boldsymbol{v}_2 = \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \text{ (component of } \boldsymbol{u} \text{ orthogonal to } \boldsymbol{a})$$

The projection theorem suggests a way to decompose a vector $\boldsymbol{u}$ into two components: one is in the same direction as $\boldsymbol{u}$ and the other is normal to $\boldsymbol{u}$.

**Example 3.4** (a) Let $\boldsymbol{u} = (2, 1, -3)$ and $\boldsymbol{e}_1 = (1, 0, 0)$ be standard unit vector. Then, the projection of $\boldsymbol{u}$ onto $\boldsymbol{e}_1$ is the vector with only *x*-component of $\boldsymbol{u}$: *i.e.*

$$proj_{\boldsymbol{e}_1}(\boldsymbol{u}) = (2, 0, 0)$$

(b) Let $\boldsymbol{u} = (2, 1, -3)$ and $\boldsymbol{a} = (4, -1, 2)$. The vector component of $\boldsymbol{u}$ along $\boldsymbol{a}$ is

$$proj_{\boldsymbol{a}}(\boldsymbol{u}) = \frac{2\cdot 4 + 1\cdot(-1) + (-3)\cdot 2}{4^2 + (-1)^2 + 2^2}(4, -1, 2) = \frac{1}{21}(4, -1, 2)$$

while the component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$ is

$$\boldsymbol{u} - proj_{\boldsymbol{a}}(\boldsymbol{u}) = \frac{1}{21}(38, 22, -65).$$

## 3.3 Lines and planes in $\mathbb{R}^n$

Consider the line $ax + by + c = 0$ in $\mathbb{R}^2$. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two distinct points on the line, as depicted in Figure 3.7(a). Then, we have

$$a(x_2 - x_1) + b(y_2 - y_1) = (a, b)\cdot(x_2 - x_1, y_2 - y_1) = 0.$$

Since the vector $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$ runs along the line, we deduce that the vector $\boldsymbol{n} = (a, b)$ is orthogonal to the line and is called *normal vector* to the line. This relation is depicted in Figure 3.7(b) and (c).

In the $xy$-plane (equivalently, $\mathbb{R}^2$), the general form of the equation of a line is $ax + by + c = 0$. When $c = 0$, the line pass through the origin. We put $\boldsymbol{v} = (x, y) = t\boldsymbol{d}$, then the line equation can be written as either

(3.13a) $\boldsymbol{n}\cdot\boldsymbol{v} = 0$ (*normal form*)

or

(3.13b) $\boldsymbol{v} = t\boldsymbol{d}$ (*parametric form with parameter,* $t$)

where $\boldsymbol{n}$ is the *normal vector* orthogonal to $\boldsymbol{v}$ (*i.e.* to the line), and $\boldsymbol{d}$ is a direction vector of the line. For the line $\ell: ax + by + c = 0$, the normal vector is given by $\boldsymbol{n} = (a, b)$. Note that the normal vector and the direction vector must be perpendicular.

If $c \neq 0$, the vector $\boldsymbol{v}$ may not be easy to define. In this case, we choose one point on the line, say $P$ as in Figure 3.8(c), to define the direction vector, $\boldsymbol{v}-\boldsymbol{p}$. The line equation becomes either

(3.14a) $\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{p})=0$

or

(3.14b) $\boldsymbol{v}-\boldsymbol{p}=t\boldsymbol{d}$.

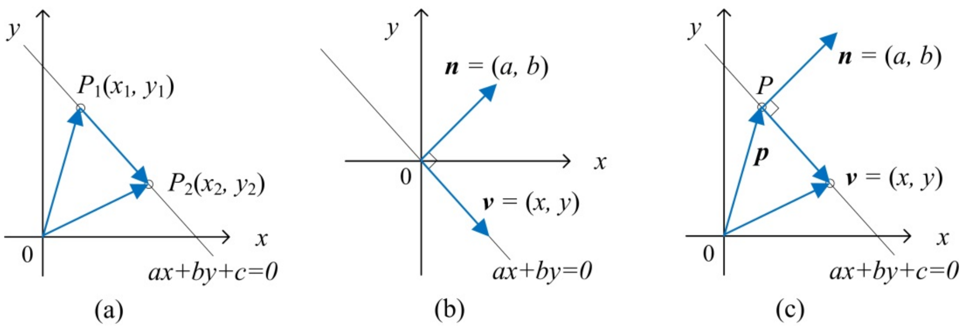


Figure 3.8 Lines in 2-dimensional Euclidean vector space

**Example 3.5** (a) Given $2x+y=0$, put $\boldsymbol{v}=(x,y)$ be a general point on the line. Then, equation for this line is given by either

$\boldsymbol{v}\cdot\boldsymbol{n}=0$, where $\boldsymbol{n}=(2,1)$, or

$\boldsymbol{v}=t\boldsymbol{d}$, where $\boldsymbol{d}=(1,-2)$ and $t$ is any scalar.

(b) The line, $2x+y=5$, has the same normal vector $\boldsymbol{n}$ as well as the direction vector $\boldsymbol{d}$. Since this line does not pass through the origin, we choose a point on the line, say $P\colon(1,3)$, corresponding to the vector $\boldsymbol{p}=(1,3)$. Then, the vector $\boldsymbol{v}-\boldsymbol{p}$ is parallel to the line and normal to $\boldsymbol{n}$. Therefore, equation for this line is given by either

$(\boldsymbol{v}-\boldsymbol{p})\cdot\boldsymbol{n}=0$ or $\boldsymbol{v}-\boldsymbol{p}=t\boldsymbol{d}$.

(c) Suppose we want to find the equation of a line in $\mathbb{R}^3$ passing through the point $P=(1,2,-1)$ and parallel to the vector $\boldsymbol{d}=(5,-1,3)$. The parametric representation can be used to represent the line as

$\boldsymbol{v}-(1,2,-1)=t(5,-1,3)$.

(d) Find a line in $\mathbb{R}^3$ through two points $P=(-1,5,0)$ and $Q=(2,1,1)$. We can choose the direction vector $\boldsymbol{d}$ as $\boldsymbol{d}=\overrightarrow{PQ}=(3,-4,1)$.

**Example 3.6** Find the distance from the point $B\colon(1,0,2)$ to the line $\ell$ through the point $A\colon(3,1,1)$ with the direction vector $\boldsymbol{d}=(-1,1,0)$.

We need to calculate the length of $\overrightarrow{PB}$ (see Figure 3.9). Let $\boldsymbol{u}=\overrightarrow{AB}$, then $\overrightarrow{AP}=proj_{\boldsymbol{d}}\boldsymbol{u}$ and $\overrightarrow{PB}=\boldsymbol{u}-proj_{\boldsymbol{d}}\boldsymbol{u}$.

(1) $\boldsymbol{u}=\overrightarrow{AB}=(1,0,2)-(3,1,1)=(-2,-1,1)$

(2) $proj_{\boldsymbol{d}}\boldsymbol{v}=\left(\frac{\boldsymbol{d}\cdot\boldsymbol{u}}{\boldsymbol{d}\cdot\boldsymbol{d}}\right)\boldsymbol{d}=\frac{1}{2}(-1,1,0)$

(3) $\boldsymbol{u} - proj_{\boldsymbol{d}}\boldsymbol{v} = \frac{1}{2}(-3, -3, 2)$

(4) The length is $\|\boldsymbol{u} - proj_{\boldsymbol{d}}\boldsymbol{v}\| = \frac{1}{2}\sqrt{(-3)^2 + (-3)^2 + 2^2} = \frac{1}{2}\sqrt{22}$

In $\mathbb{R}^2$, the distance between a point $B: (x_0, y_0)$ and the line $\ell :\ ax + by + c = 0$ is given by

(3.16) $$d(B, \ell) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

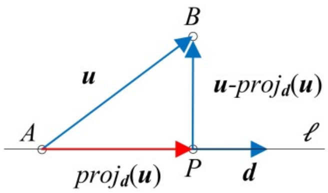


Figure 3.9 Distance from a point to a line

In *xyz*-plane (equivalently, in $\mathbb{R}^3$), the general form of the equation of a plane is $P :\ ax + by + cz + d = 0$. We might reasonably guess that, if $ax + by + c = 0$ is the general form of the equation of a line in $\mathbb{R}^2$, then $ax + by + cz + d = 0$ might also represent a line in $\mathbb{R}^3$.

If $d = 0$, then the plane equation in normal form can be written by $\boldsymbol{n} \cdot \boldsymbol{v} = 0$, where $\boldsymbol{n}$ is the normal vector and is given by $\boldsymbol{n} = (a, b, c)$**.** However, the set of all vectors $\boldsymbol{v}$ that satisfies this equation is the set of all vectors orthogonal to $\boldsymbol{n}$. As shown in Figure 3.10(a), vectors in infinitely many directions have this property, determining a family of parallel planes. For example, consider $ax = 0$. The normal vector to this equation is $\boldsymbol{n} = (a, 0, 0)$, so that all vectors in $yz$-plane is orthogonal to $\boldsymbol{n}$. Therefore, $ax + by + cz + d = 0$ represents a plane, not a line in $\mathbb{R}^3$.

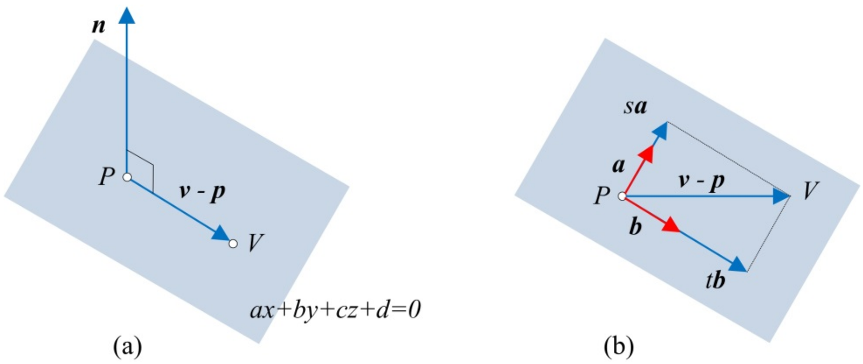


Figure 3.10 Place in $\mathbb{R}^3$

Similarly to the line equation, the plane equation can be described as either

(3.17) $$\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{p}) = 0$$

or

(3.18) $$\boldsymbol{v} = \boldsymbol{p} + s\boldsymbol{a} + t\boldsymbol{b}$$

where $\boldsymbol{p}$ is vector of a point $P$ on the plane and $\boldsymbol{a}$ and $\boldsymbol{b}$ are direction vectors for the plane.

**Example 3.7** Find the vector form of the plane in $\mathbb{R}^3$ through the point $P: (6, 0, 1)$ with normal vector $\boldsymbol{n} = (1, 2, 3)$.

Choose $\boldsymbol{p} = (6, 0, 1)$ and $\boldsymbol{v} = (x, y, z)$. Then, the equation for the plane is given by

$$(1, 2, 3) \cdot (x - 6, y, z - 1) = x - 6 + 2y + 3(z - 1) = x + 2y + 3z - 9 = 0$$

In order to find the parametric equation, we need two direction vectors of the plane. Choose any two points on the plane: say $Q: (9, 0, 0)$ and $R: (3, 3, 0)$. Let two direction vectors be

$$\boldsymbol{a} = \overrightarrow{PQ} = (3, 0, -1) \text{ and } \boldsymbol{b} = \overrightarrow{PR} = (-3, 3, -1)$$

then the parametric equation is given by

$$(x, y, z) = (6, 0, 1) + s(3, 0, -1) + t(-3, 3, -1)$$

or

$$x = 6 + 3s - 3t$$

$$y = 3t$$

$$z = 1 - s - t$$