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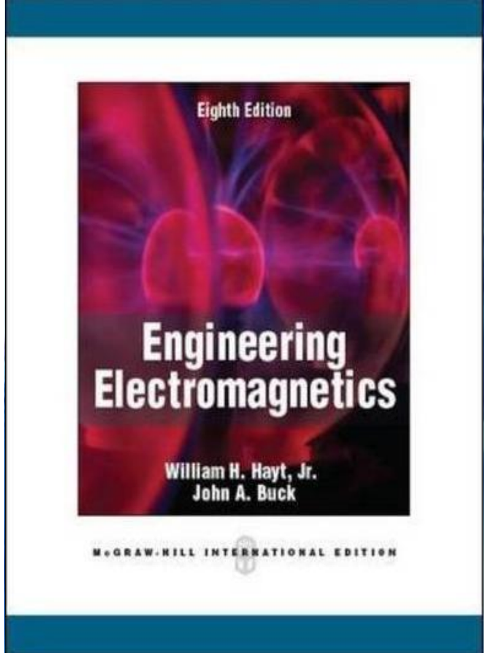

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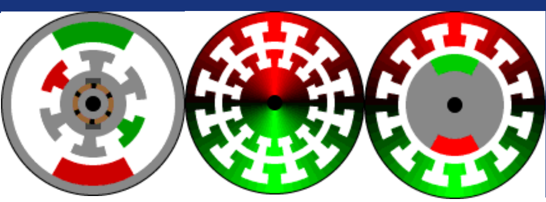

Engineering Electromagnetics (Week 12)

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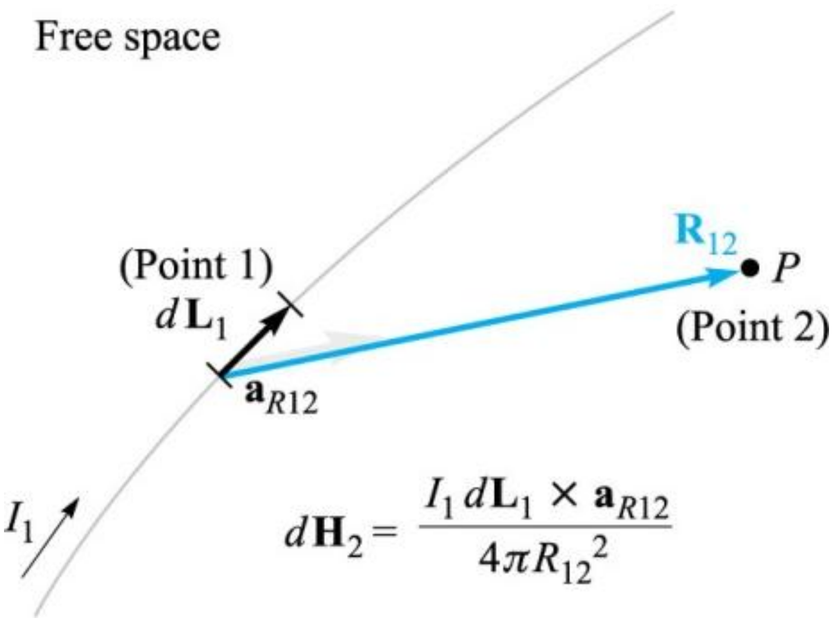
Ch.7 The Steady Magnetic Field

7.1 BIOT-SAVART LAW

- The Source of the Steady Magnetic Field
  1. A Permanent Magnet
  2. An Electric field Changing Linearly with Time
  3. A Differential DC Element



Biot-Savart Law

$$d\mathbf{H} = \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I d\mathbf{L} \times \mathbf{R}}{4\pi R^3} \quad [A/m]$$

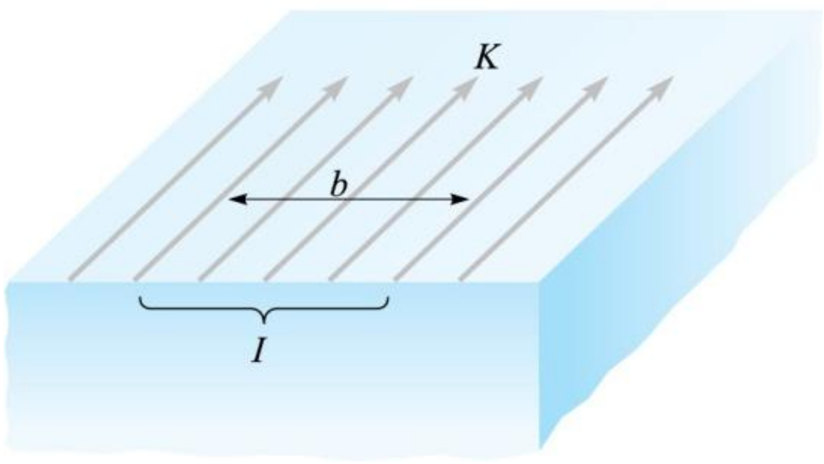
- $d\mathbf{H}$ is proportional to current and $d\mathbf{L}$
- Inversely proportional to R^2
- Direction is $(d\mathbf{L}_1 \times \mathbf{a}_{R12})$
- Similar to Coulomb's Law

$$cf)\ d\mathbf{E}_2 = \frac{dQ_1 \times \mathbf{a}_{R12}}{4\pi\epsilon_0 {R_{12}}^2}$$

- The total field arising from the closed circuit path,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

- Two- and three-dimensional currents



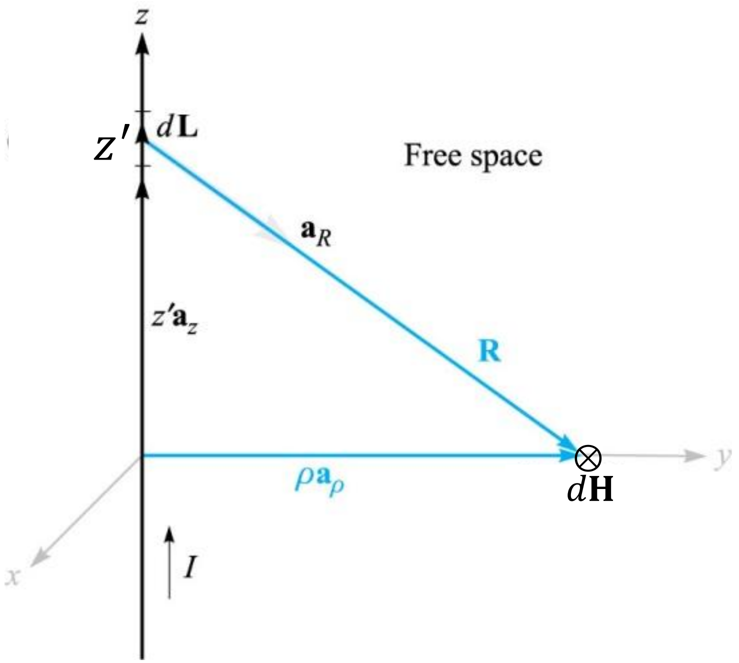
$I = Kb$ (K:Surface current density)

$I d\mathbf{L} = \mathbf{K} dS = \mathbf{J} dv$

$$\therefore \mathbf{H} = \int_S \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2} \quad \text{: Two−dimension}$$

$$\therefore \mathbf{H} = \int_{vol} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2} \quad \text{: Three−dimension}$$

□ Example : Biot-Savart Law in free space



$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

$$\mathbf{a}_R = \frac{\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

Unit field intensity will be,

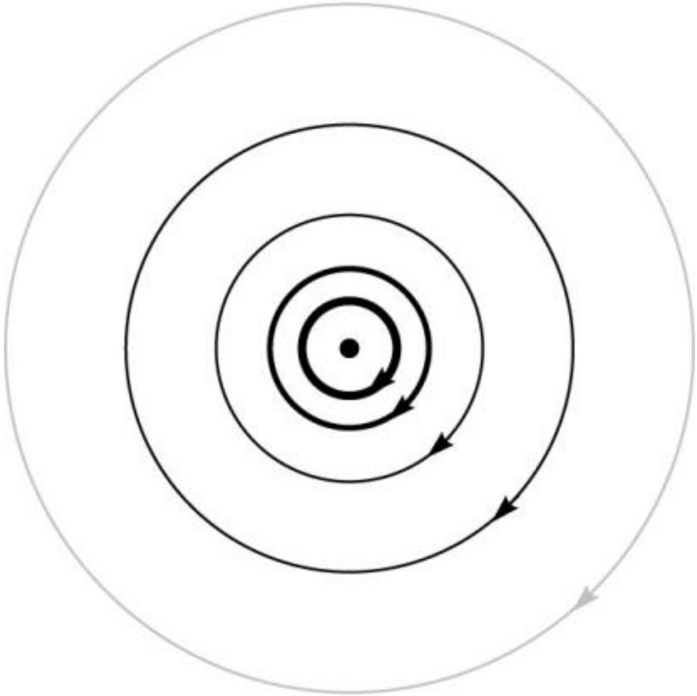
$$d\mathbf{H} = \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{Idz' a_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Integrate it over the entire wire,

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{Idz' a_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$\therefore\ \mathbf{H} = \frac{I}{4\pi}\int_{-\infty}^{\infty} \frac{\rho dz' \mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

□ Example : Biot-Savart Law in free space (cont.)



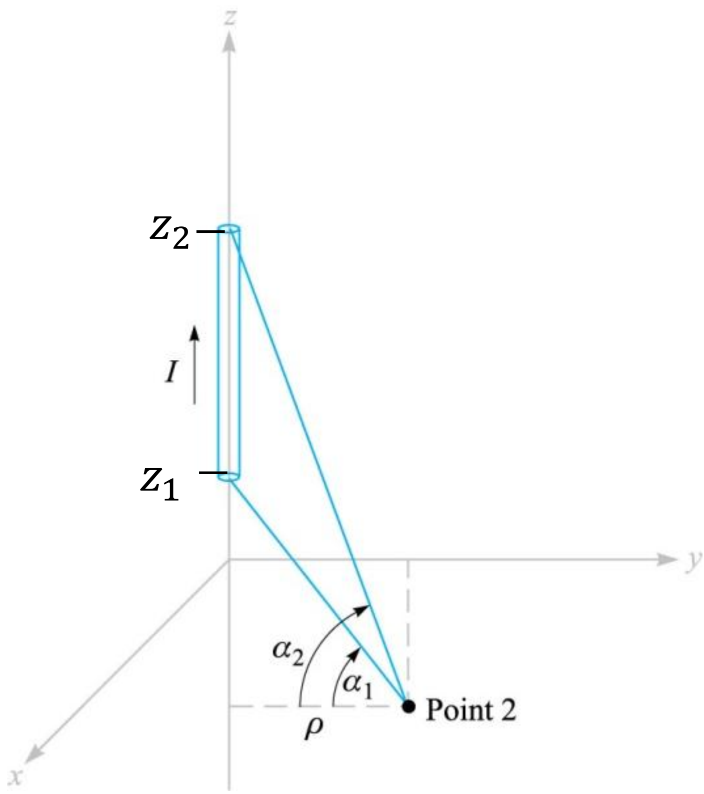
$$\mathbf{H} = \frac{I}{4\pi}\int_{-\infty}^{\infty}\frac{\rho dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I\rho\mathbf{a}_\phi}{4\pi}\frac{z'}{\rho^2\sqrt{\rho^2 + z'^2}}\Bigg|_{-\infty}^{\infty}$$

$$\therefore\ \mathbf{H} = \frac{1}{2\pi\rho}\mathbf{a}_\phi$$

- Magnetic field streamlines are concentric circles, whose magnitudes decrease as the inverse distance from the z axis

- Example : Magnetic Field from a finite current segment

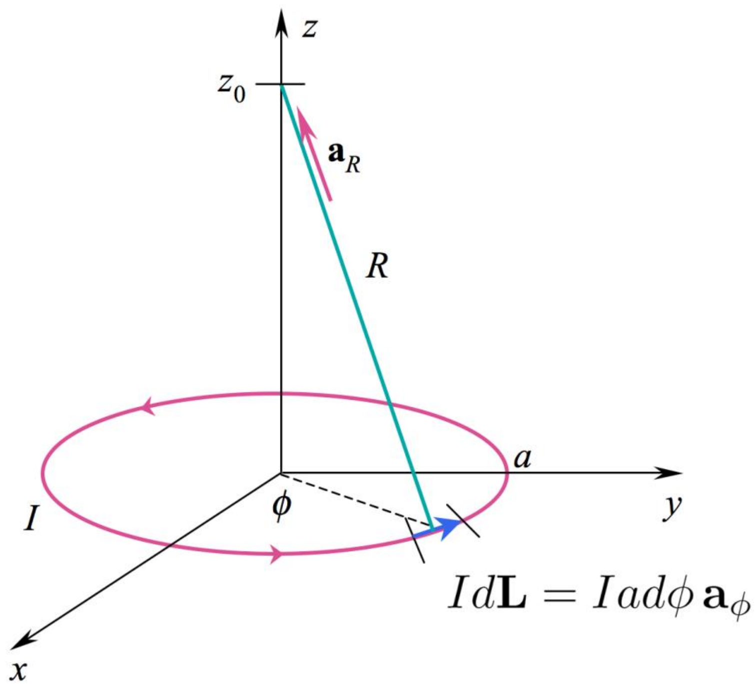


$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$$= \int_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} \frac{I dz' a_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$\therefore \quad \mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$

□ Example : Magnetic Field from a current loop

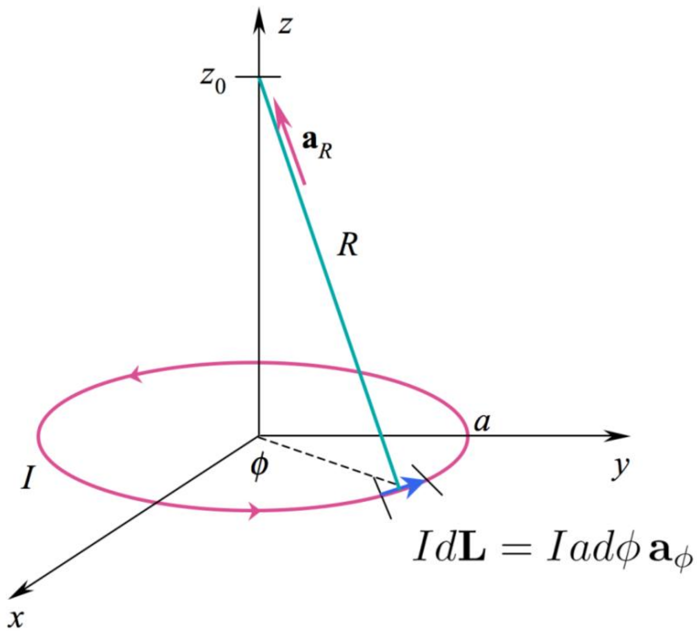


$$\mathbf{H} = \int \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$$\begin{cases} Id\mathbf{L} = Iad\phi\mathbf{a}_\phi \\ R = \sqrt{a^2 + z_0^2} \\ \mathbf{a}_R = \dfrac{z_0\mathbf{a}_z - a\mathbf{a}_\rho}{\sqrt{a^2 + z_0^2}} \end{cases}$$

$$\therefore\ \mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\mathbf{a}_\phi \times (z_0\mathbf{a}_z - a\mathbf{a}_\rho)}{4\pi(a^2 + z_0^2)^{3/2}}$$

- Example : Magnetic Field from a current loop (cont.)



$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\boldsymbol{a}_\phi \times (z_0\mathbf{a}_z - a\mathbf{a}_\rho)}{4\pi(a^2+z_0^2)^{3/2}}$$

$$= \int_0^{2\pi} \frac{Iad\phi\,(z_0\mathbf{a}_\rho + a\mathbf{a}_z)}{4\pi(a^2+z_0^2)^{3/2}}$$

Include the angle dependence in the radial unit vector,

$$\mathbf{a}_\rho = \cos\phi\mathbf{a}_\rho + \sin\phi\mathbf{a}_\rho$$

$$\therefore\ \mathbf{H} = \frac{I(\pi a^2)\boldsymbol{a}_z}{2\pi(a^2+z_0^2)^{3/2}}$$

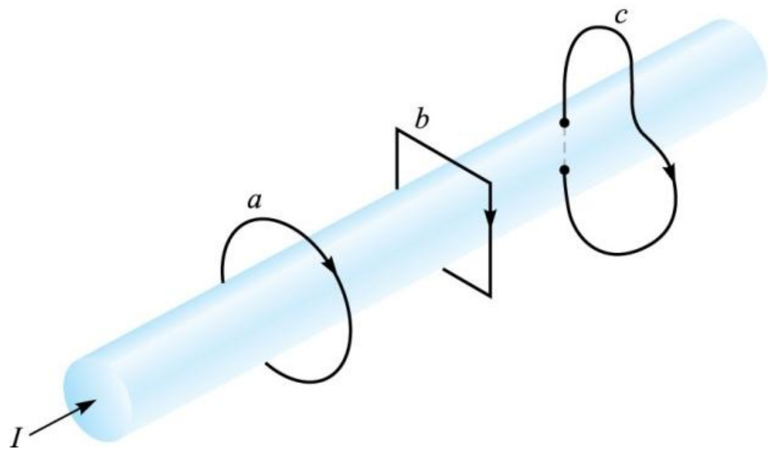
♯ Magnetic moment

$$\mathbf{m} = I(\pi a^2)\boldsymbol{a}_z$$

7.2 AMPERE'S CIRCUITAL LAW

- Ampere's Circuital Law

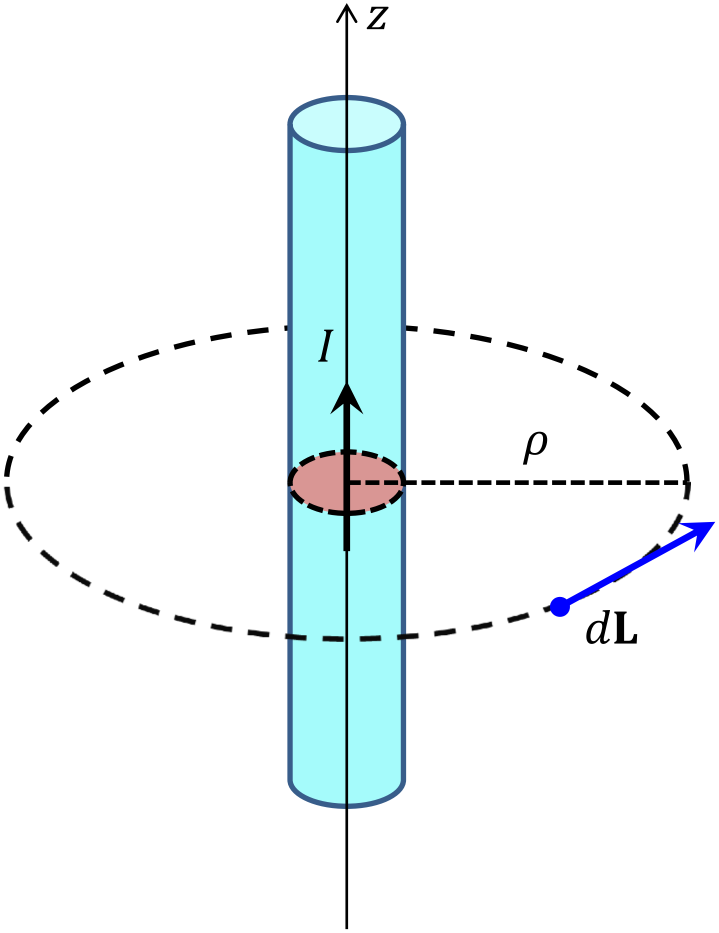
"The line integral of H about any closed path is exactly equal to the direct current enclosed by that path"



$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

In the figure ,the integral of **H** about closed paths a and b gives the total current I, while the integral over path c gives only that portion of the current that lies within c

□ Ampere's Law Applied to a Long Wire



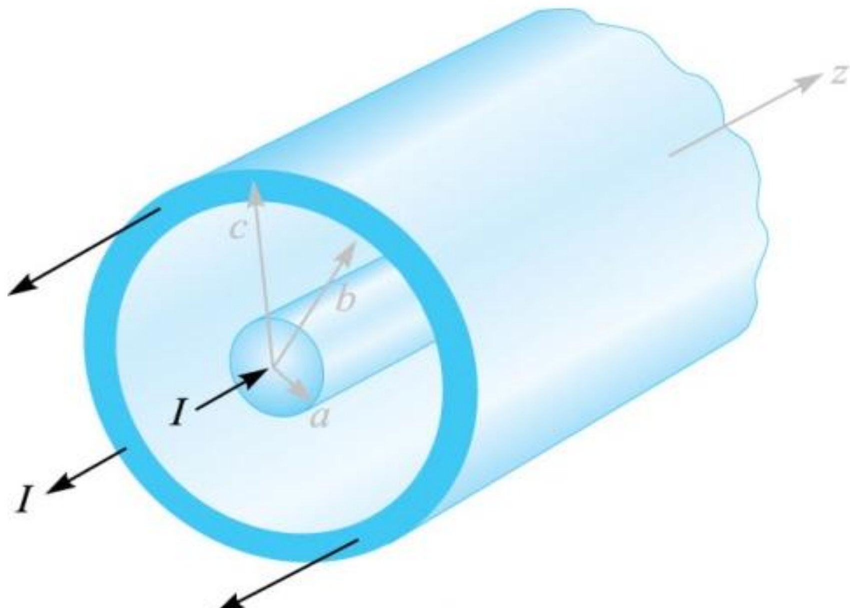
1. The Current Flows in the direction of $\mathbf{a}_z$
2. Symmetry suggests that **H** will be circular, constant-valued at constant radius

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi$$

$$= H_\phi 2\pi\rho = I$$

$$\therefore \; H_\phi = \frac{1}{2\pi\rho} \quad or \quad \mathbf{H} = \frac{1}{2\pi\rho}\mathbf{a}_\phi$$

- Coaxial Transmission Line



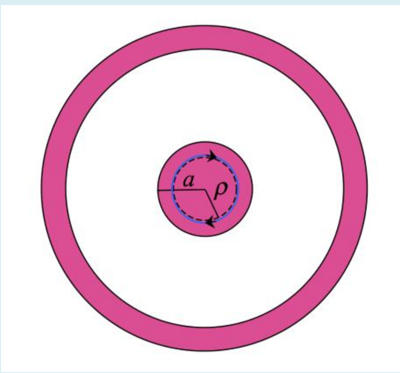
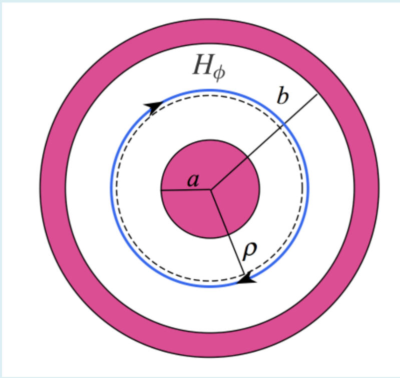
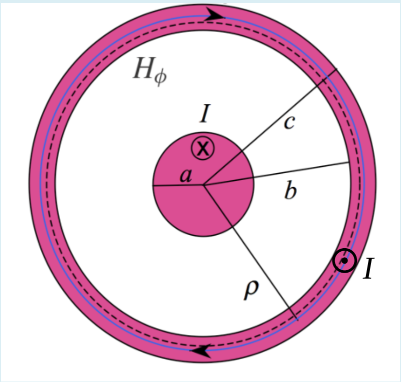
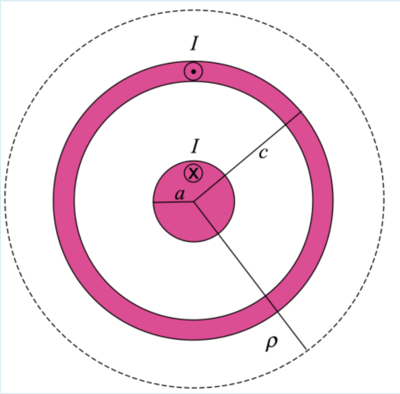
(a)

4 cases can be considered (ρ : radius) :

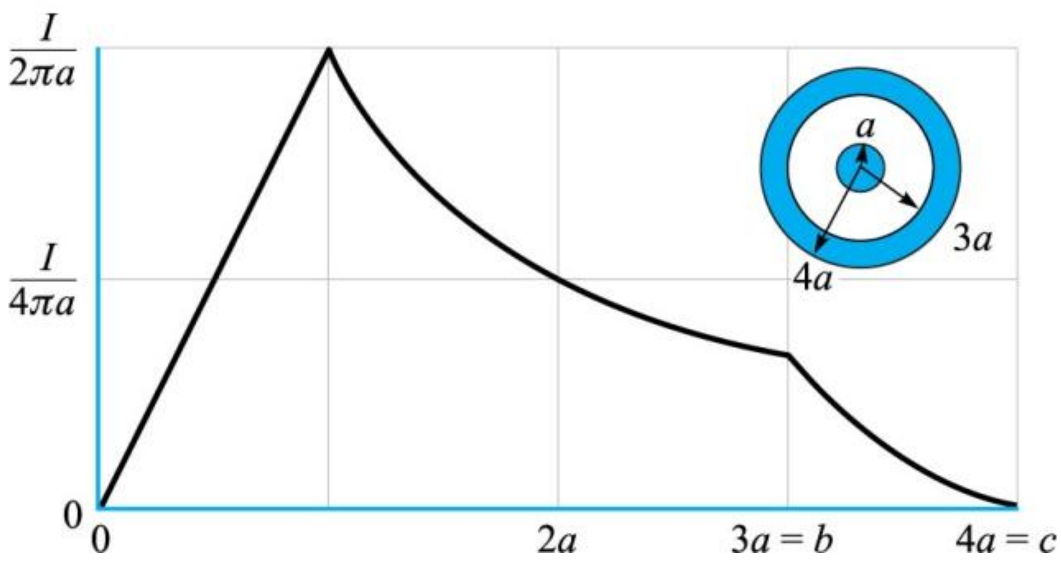
1. $\rho < a$
2. $a < \rho < b$
3. $b < \rho < c$
4. $\rho > c$

7.2 AMPERE'S CIRCUITAL LAW

□ Coaxial Transmission Line (cont.)

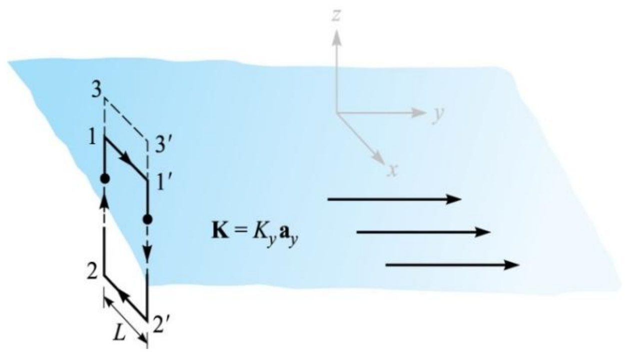
$\rho < a$	$a < \rho < b$	$\mathrm{b} < \rho < c$	$\rho > c$
			
$\oint \mathbf{H}\cdot d\mathbf{L} = H_\phi 2\pi\rho = I_{encl}$ $I_{encl} = I\dfrac{\rho^2}{a^2}$ $2\pi\rho H_\phi = I\dfrac{\rho^2}{a^2}$ $\therefore H_\phi = \dfrac{I\rho}{2\pi a^2}$	$\oint \mathbf{H}\cdot d\mathbf{L} = H_\phi 2\pi\rho = I_{encl}$ $I_{encl} = I$ $\therefore H_\phi = \dfrac{I}{2\pi\rho}$	$\oint \mathbf{H}\cdot d\mathbf{L} = H_\phi 2\pi\rho = I_{encl}$ $I_{encl} = I - I\left(\dfrac{\rho^2 - b^2}{c^2 - b^2}\right)$ $\therefore H_\phi = \dfrac{I}{2\pi\rho}\left(\dfrac{c^2 - \rho^2}{c^2 - b^2}\right)$	$\oint \mathbf{H}\cdot d\mathbf{L} = H_\phi 2\pi\rho = I$ $I_{encl} = I - I = 0$ $\therefore H_\phi = \dfrac{I}{2\pi\rho}$

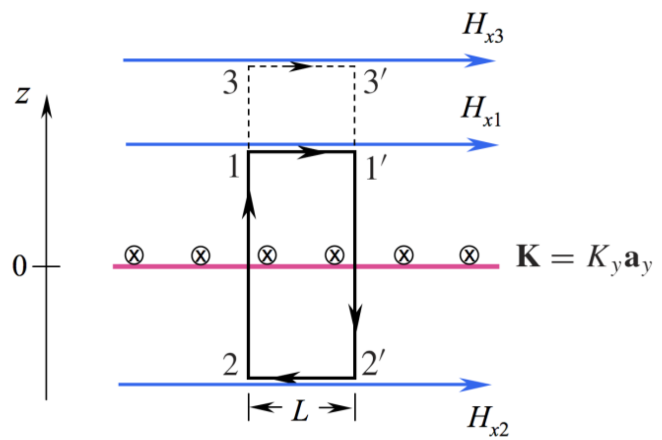
- Coaxial Transmission Line (cont.)



1. Magnetic field is continuous on interface of conductor
2. Ideally, external magnetic field is zero : shielding
3. External adjacent circuit does not affected by even large current flows in coaxial cable.

- Infinite Plane Current



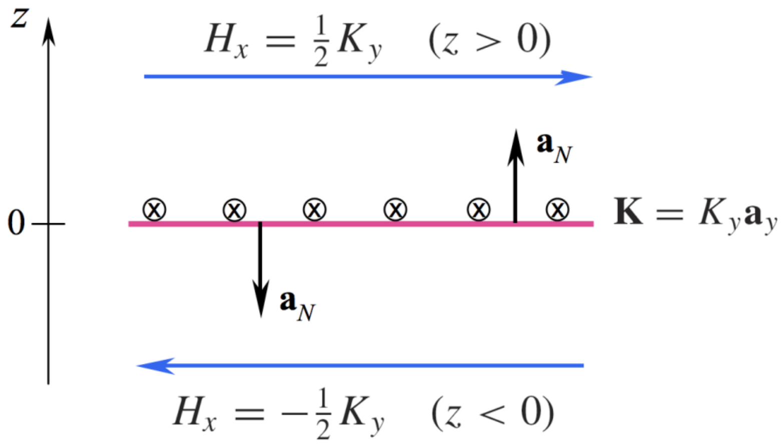


(K: surface current density, $K_y\mathbf{a}_y$)

1. Ampere's circuital law
   - Paths 1-1'-2'-2-1 : $H_{x1}L + H_{x2}(-L) = K_yL$ or $H_{x1} - H_{x2} = K_y$
   - Paths 3-3'-2'-2-3 : $H_{x3}L + H_{x2}(-L) = K_yL$ or $H_{x3} - H_{x2} = K_y$
2. According to 1-(1) and 2-(1),
   - $H_{x1} = H_{x3}$ ➔ Constant field in each region (above and below the current plane)
3. By symmetry, $\begin{cases} H_x = \frac{1}{2}K_y & (z > 0) \\ H_x = -\frac{1}{2}K_y & (z < 0) \end{cases}$

- Infinite Plane Current (cont.)

  The actual field configuration is shown below,



$$\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_N$$

($\mathbf{a}_N$: the unit vector that is normal to the current sheet)

□ Infinite Plane Current (Second sheet added)

1. $z > \frac{d}{2}, z < -\frac{d}{2}$

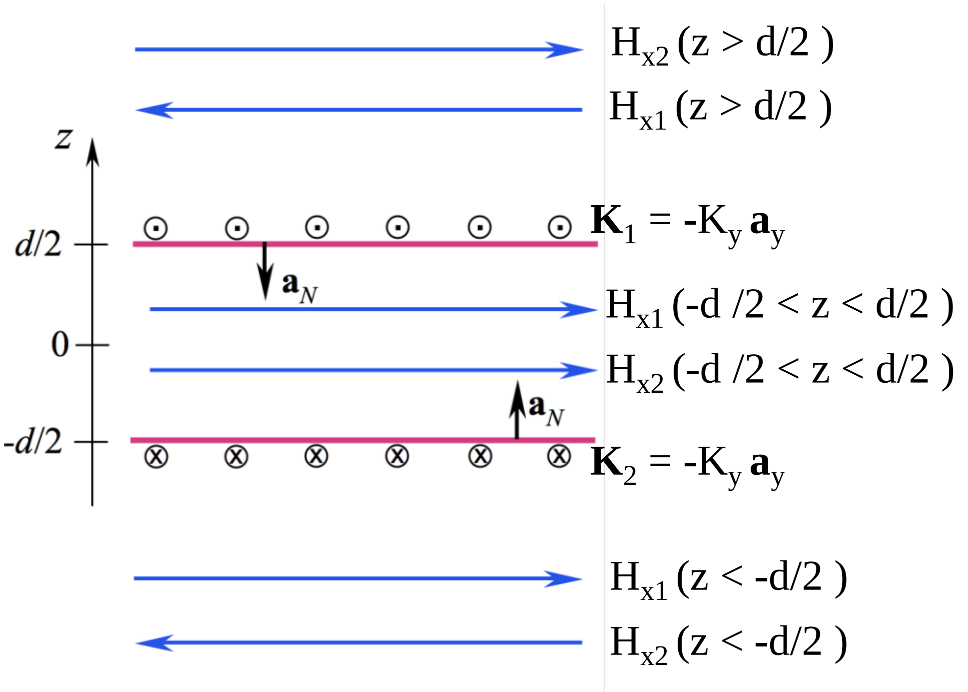
   H_{x1} and H_{x2} cancel out to,

   $$\mathbf{H} = 0$$

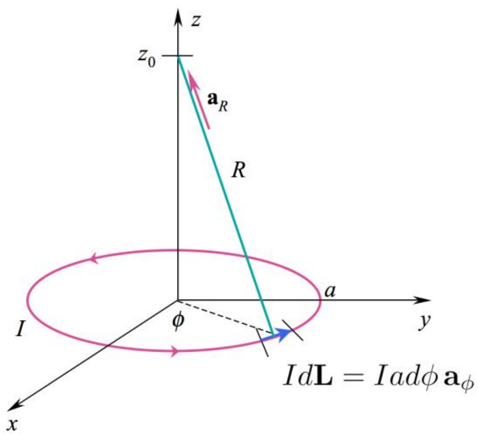
2. $-\frac{d}{2} < z < \frac{d}{2}$

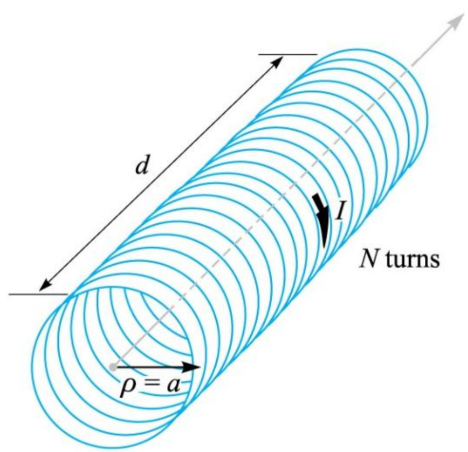
   H_{x1} and H_{x2} add to give,

   $$\mathbf{H} = \mathbf{K} \times a_N$$



- Solenoid
  1. Interpretation using Biot-Savart Law





Assumed to have many tightly-wound turns

1) Magnetic field for single loop

$$\mathbf{H} = \frac{I(\pi a^2)a_z}{2\pi(a^2 + z_0^2)^{3/2}}$$

2) Solenoid Magnetic field

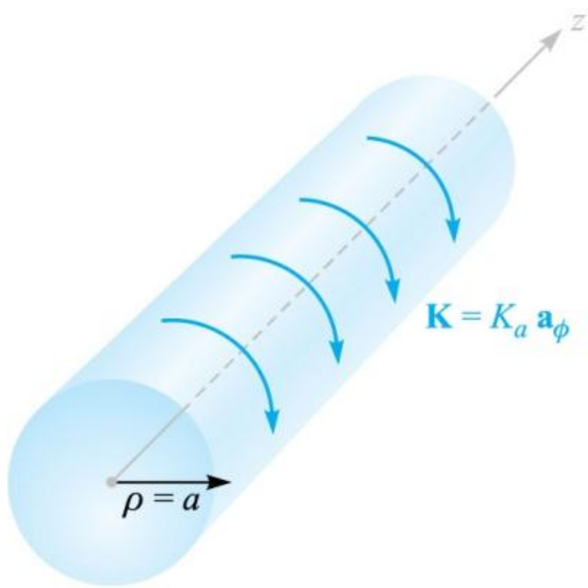
$$dI = \frac{d}{N} Idz$$ therefore, (1) became,

$$d\mathbf{H} = \frac{(N/d)Idz(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z^2)^{3/2}}$$

$$H = \int dH = \int_{-d/2}^{d/2} \frac{(N/d)Idz(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z^2)^{3/2}}$$

$$= \frac{NI\mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}} \Rightarrow \mathbf{H} \cong \frac{NI}{d}\mathbf{a}_z (d \gg a)$$

- Solenoid (cont.)
  2. Interpretation using surface current
     1) On-axis field



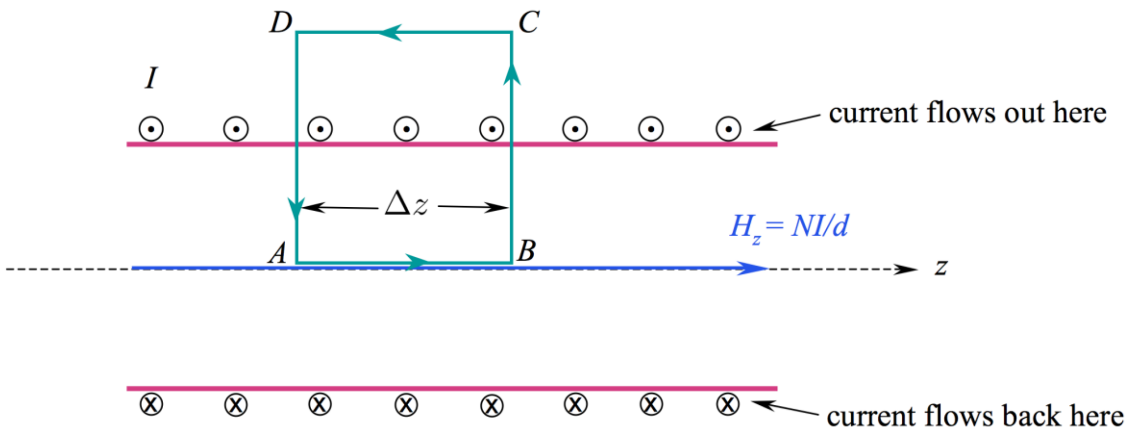
$$K = K_a a_\phi$$
$$= \frac{NI}{d} a_\phi \qquad \text{A/m}$$

Therefore,

$$\mathbf{H}(\rho = z = 0) = \frac{K_a \mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}}$$
$$\cong K_a \mathbf{a}_z (d \gg a) \qquad \text{A/m}$$

The on-axis field magnitude near the center of a cylindrical current sheet, where current circulates around the z axis, and whose length is much greater than its radius, is just the surface current density.

- Solenoid (cont.)
  2. Interpretation using surface current
     2) Off-axis field

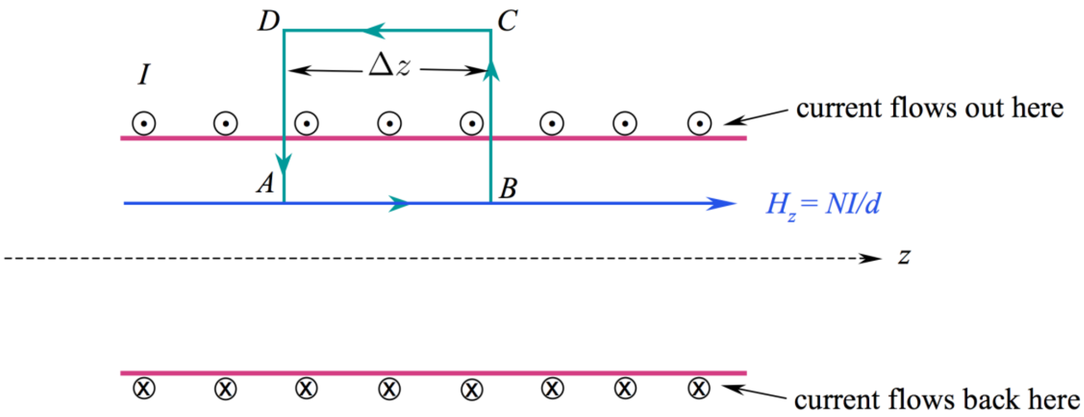


$$\oint H \cdot dL = \int_A^B H_z dz + \int_B^C H_\rho d\rho + \int_C^D H_{z,out} dz + \int_D^A H_\rho d\rho = I_{encl} = \frac{NI}{d}\Delta z$$

$\frac{NI}{d}\Delta z$(on-axis field)

Radial Path

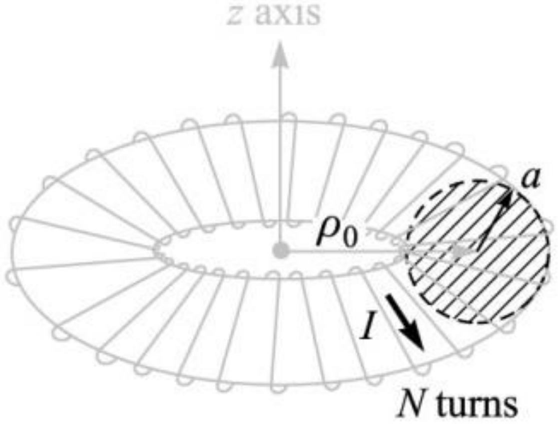
- Solenoid (cont.)
  2. Interpretation using surface current
     2) Off-axis field

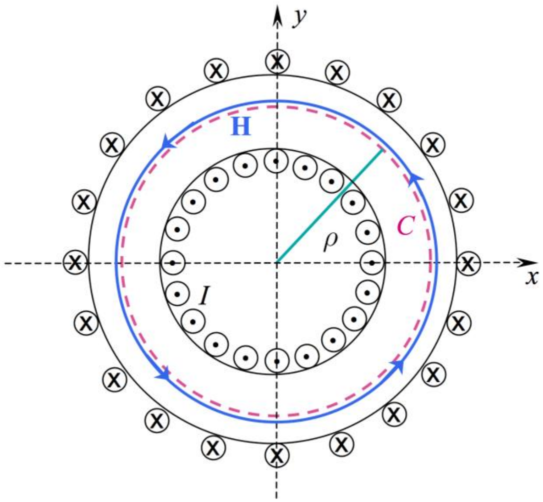


The situation does not change if the lower z-directed path is raised above the z axis. The vertical paths still cancel, and the outside field is still zero. The field along the path A to B is therefore NI/d as before.

The magnetic field within a long solenoid is approximately constant throughout the coil cross-section, and is H_z = NI/d.

□ Toroid





1. Inside of toroid $(\rho_0 - a < \rho < \rho_0 + a)$

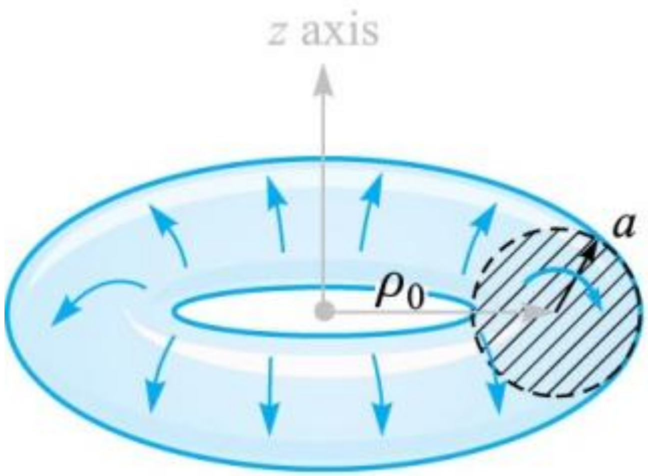
$$\oint_C H \cdot dL = 2\pi\rho H_\phi = I_{encl} = NI$$

$$\therefore \quad H_\phi = \frac{NI}{2\pi\rho}$$

2. Outside of toroid $(\rho < \rho_0 - a,\ \rho > \rho_0 + a)$

Zero magnetic field (∵ no enclosed current)

♯ Cf. Magnetic Field in Toroid using Surface Current Density



1. Inside of toroid ($\rho_0 - a < \rho < \rho_0 + a$)

$$\oint_C H \cdot dL = 2\pi\rho H_\phi = I_{encl} = 2\pi(\rho_0 - a)K_a$$

$$\therefore \quad H_\phi = \frac{\rho_0 - a}{\rho} K_a$$

2. Outside of toroid ($\rho < \rho_0 - a,\ \rho > \rho_0 + a$)

Zero magnetic field (∵ no enclosed current)

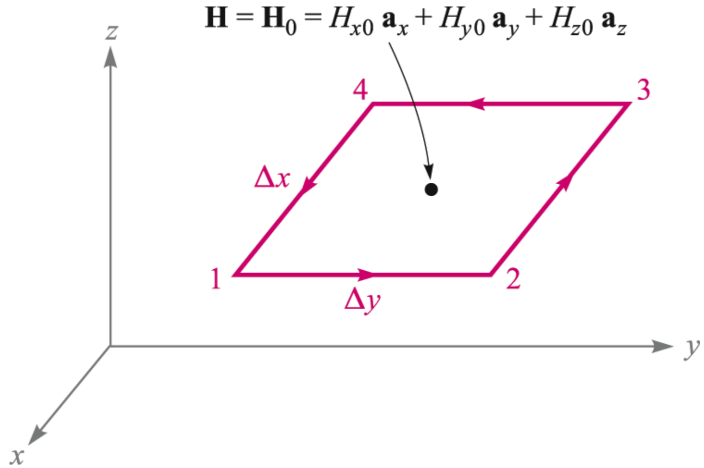
- In differential closed loop in xy-plane,

  Along path 1-2,

$$(H \cdot \triangle L)_{1-2} = H_{y,1-2} \triangle y$$

$$H_{y,1-2} \triangle y \simeq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x} \triangle x \right) \triangle y$$

$$\therefore \quad (H \cdot \triangle L)_{1-2} \simeq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x} \triangle x \right) \triangle y$$



  In the same method,

$$\therefore \begin{cases} (H \cdot \triangle L)_{2-3} \simeq -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y} \triangle y \right) \triangle x \\ (H \cdot \triangle L)_{3-4} \simeq -\left(H_{y0} - \frac{1}{2}\frac{\partial H_y}{\partial x} \triangle x \right) \triangle y \\ (H \cdot \triangle L)_{4-1} \simeq \left(H_{x0} - \frac{1}{2}\frac{\partial H_x}{\partial y} \triangle y \right) \triangle x \end{cases}$$

∴ For Closed loop (sum of all path)

$$\oint \mathbf{H} \cdot d\mathbf{L} \cong \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \triangle x \triangle y$$

From Ampere's Law,

$$\oint \mathbf{H} \cdot d\mathbf{L} \cong \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \Delta x \Delta y \cong J_z \Delta x \Delta y$$

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \cong \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \cong J_z$$

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$: z component of current density

In the same method, closed loop in yz-plane and zx-plane,

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x, \quad \lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

- Curl
  1. Notation

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H}$$

  2. Calculation

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}$$

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

From the previous calculation,

$\nabla \times \mathbf{H} = \mathbf{J}$: Point form of Ampere's Law

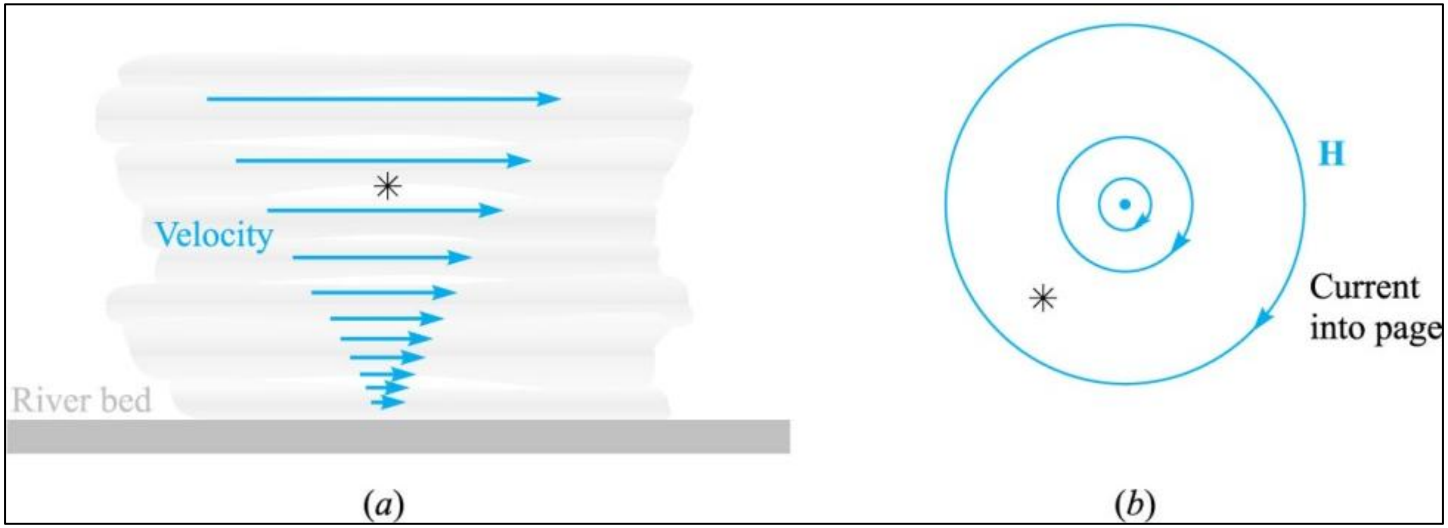
- Curl in other coordinate systems
  1. Cylindrical

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z$$

  2. Spherical

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi$$

- Visualization of curl
  - Place a small “paddle wheel” in a flowing stream
  - If the wheel rotate, vector has curl component



  - The wheel will rotate clockwise

    → A curl component will point into the screen
  - Positioning the wheel at all three orthogonal orientations will yield measurements of all three components of the curl.

7.3 CURL

- As in static magnetic field,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I \quad \Longleftrightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}$$

- Static electric field can be notated as,

$$\oint E \cdot d\mathbf{L} = 0 \quad \Longleftrightarrow \quad \nabla \times \mathbf{E} = 0$$

∴ A field is conservative if it has zero curl at all points over which the field is defined

- When $\Delta S \to 0$,

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \cong (\nabla \times \mathbf{H})_N$$

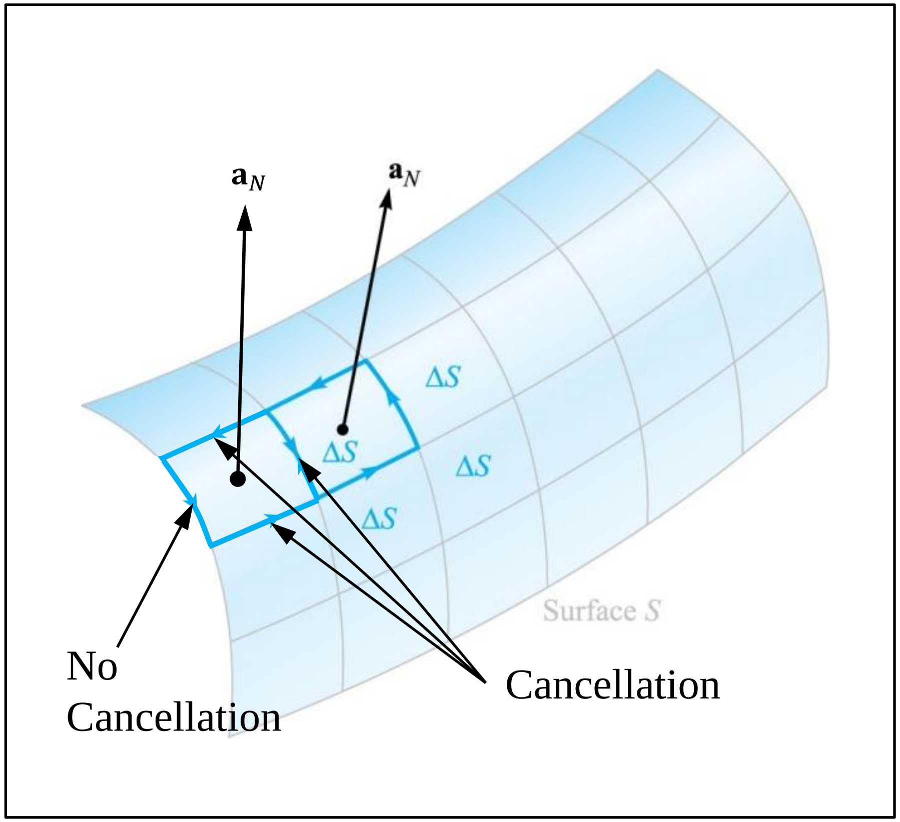
$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \cong (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \cong (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \, \Delta \mathbf{S}$$

$$= (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S}$$

Apply Cancellation then,

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$



7.4 STOKES' THEOREM

- Ampere's circuital law : Point and integral form

$$\nabla \times \mathbf{H} = \mathbf{J}$$: Point Form

$$\int_S (\nabla \times \mathbf{H}) \cdot d\,\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

$$\oint \mathbf{H} \cdot d\mathbf{L} = \boldsymbol{I}$$: Integral form

- In any vector space,

$$\nabla \cdot \nabla \times \mathbf{A} = T \qquad \text{(A: vector space)}$$

$$\int_{vol} (\nabla \cdot \nabla \times \mathbf{A}) \cdot d\,\mathbf{S} = \int_{vol} T \cdot dv$$

$$\oint_{S} (\nabla \times \mathbf{A}) \cdot dS = \int_{vol} T \cdot dv \qquad (\because \textit{divergence theorem})$$

$$\int_{vol} T \cdot dv = 0 \quad (\because \text{Integral of stokes}'\text{theorem over closed path is zero})$$

$$\therefore T = 0 \qquad \rightarrow \qquad \boxed{\nabla \cdot \nabla \times \mathbf{A} \equiv 0}$$

Therefore,

$$\nabla \cdot \nabla \times \mathbf{H} = \boxed{\nabla \cdot \mathbf{J} = 0 \qquad \text{(Current Continuity Equation)}}$$