

3.3. Solutions of Laplace Equation

The Laplace equation is an elliptic type partial differential equation which requires boundary conditions on all surrounding boundaries. The Laplace equation can be solved by such techniques as Laplace transform, separation of variables and others. Here, we use a simpler one, the separation of variables. The procedure is as follows:

1. First of all, the solution form is assumed to be periodic both in horizontal axis and time. Then

$$\Phi = \Phi(x, z, t) = \Phi(\Theta, z)$$

in which $\Theta = x - ct$ with c being the phase speed (celerity) defined as $c = \omega/k$. That is, a periodic function both in time and in x -axis is assumed. This can be viewed as lateral boundary conditions for the Laplace equation.

2. The velocity potential function can be written as

$$\Phi(\Theta, z) = G(\Theta) Z(z)$$

Then by using the separation of variables the governing equation is expressed as

$$\partial_{\Theta\Theta}\Phi + \partial_{zz}\Phi = 0$$

$$G''/G + Z''/Z = 0$$

$$G''/G = -Z''/Z = -k^2$$

3. Since G and Z are functions of $\Theta(x, t)$ and z , respectively, k is a constant and we may have

$$G'' + k^2 G = 0, \quad G = (\cos k\Theta, \sin k\Theta)$$

$$Z'' - k^2 Z = 0, \quad Z = (e^{+kz}, e^{-kz})$$

We then obtain

TABLE 3.1 Possible Solutions to the Laplace Equation, Based on Separation of Variables

Character of k , the Separation Constant	Ordinary Differential Equations	Solutions
Real	$\frac{d^2 X}{dx^2} + k^2 X = 0$	$X(x) = A \cos kx + B \sin kx$
$k^2 > 0$	$\frac{d^2 Z}{dz^2} - k^2 Z = 0$	$Z(z) = Ce^{kz} + De^{-kz}$
$k = 0$	$\frac{d^2 X}{dx^2} = 0$	$X(x) = Ax + B$
	$\frac{d^2 Z}{dz^2} = 0$	$Z(z) = Cz + D$
Imaginary		
$k^2 < 0, k = i k $	$\frac{d^2 X}{dx^2} - k ^2 X = 0$	$X(x) = Ae^{ k x} + Be^{- k x}$
$ k = \text{magnitude of } k$	$\frac{d^2 Z}{dz^2} + k ^2 Z = 0$	$Z(z) = C \cos k z + D \sin k z$

$$\Phi = GZ$$

$$= (C_1 \cos k\Theta + C_2 \sin k\Theta) \times (C_3 e^{kz} + C_4 e^{-kz})$$

in which C_1, C_2, C_3, C_4 are integral constants. From the SBC (BBC)

$$w = 0, \quad z = -h$$

Then,

$$\partial_z \Phi = k(C_1 \cos k\Theta + C_2 \sin k\Theta) \times (C_3 e^{-kh} - C_4 e^{kh}) = 0$$

4. Since $C_1 \cos k\Theta + C_2 \sin k\Theta \neq 0$, $C_3 e^{-kh} - C_4 e^{kh} = 0$ for nontrivial solutions.

Thus, we have $C_3 = C_4 e^{2kh}$ and

$$\begin{aligned} C_3 e^{kz} + C_4 e^{-kz} &= C_4 e^{kh} (e^{k(z+h)} + e^{-k(z+h)}) \\ &= C_4 e^{kh} 2 \cosh k(z+h) \\ &= C_5 \cosh k(z+h) \end{aligned}$$

in which C_5 is an integral constant. And

$$C_1 \cos k\Theta + C_2 \sin k\Theta = C_6 \sin(k\Theta + \delta)$$

5. Finally, we have

$$\Phi = C \cosh k(z+h) \times \sin(kx - \omega t)$$

in which we set the phase shift, δ , at zero without loss of generality and

C is still a constant to be determined. The function Θ is defined as

$\Theta = x - ct$ and the phase function S is also defined as $S = kx - \omega t = k\Theta$.

6. To determine the constant C we first define the free surface displacement as

$$\zeta = a \cos(kx - \omega t)$$

Now, we can use DFSBC to determine the unknown constant C .

7. The constant is then determined as $C = ga / (\omega \cosh kh)$. Finally, we can have the velocity potential function given by

$$\Phi = \frac{ga}{\omega} \frac{\cosh k(z+h)}{\cosh kh} \sin(kx - \omega t)$$

This is the velocity potential function which satisfies all boundary conditions and this velocity potential function also implies that water particles can move due to wave motion.

3.4. Dispersion Relation

3.4.1. dispersion relation

- a. To derive the dispersion relation we can use the CFSBC, that is

$$\partial_{tt}\Phi + gw = 0, \quad z = 0$$

By substituting the velocity potential function into the CFSBC, we obtain

$$\omega^2 = gk \tanh kh$$

in which $\tanh kh$ can be expanded by Taylor series if kh is small enough, that is

$$\tanh kh \approx kh - (kh)^3/3 + \dots$$

- b. shallow water depth limit, that is $kh \rightarrow 0$

$$\omega^2 \approx gk^2 h$$

$$c^2 = gh$$

Thus, the phase speed is no longer dependent on the wavenumber (or wavelength). This type wave is called a nondispersive wave. In very shallow water, the wave speed is only a function of water depth. This is already discussed in hydraulics.

- c. deep water depth limit, that is $kh \rightarrow \infty$ ($\tanh kh \rightarrow 1$ as $kh \rightarrow \infty$)

$$\omega^2 \approx gk$$

$$c^2 = g/k$$

Thus, the phase speed is dependent on the wavenumber (or wavelength). This type wave is called a dispersive wave. In very deep water, the wave speed is nothing to do with water depth. The phase speed is only a function of wavelength.

d. It is remarked that it is still not well defined the limits for both shallow and deep water depths. In coastal engineering, however, the shallow water depth is generally defined as $kh < 0.1\pi$, whereas the deep water depth is defined as $kh > \pi$. The water depth between the shallow and the deep water depths is called intermediate water depth. In spite of these criteria, you should depend on your experience to judge a water depth in many practical and theoretical problems.

e. physical meaning of the dispersion relation

As discussed previously, the wave speed can be calculated from the dispersion relation, that is

$$\omega^2 = gk \tanh kh, \quad c = \left(\frac{g}{k} \tanh kh \right)^{1/2}$$

As seen in the dispersion relation the longer wave can propagate faster than the shorter wave. This is the physical meaning of the dispersion relation.