3.4 Heine-Borel Theorem, part 2

First of all, let us summarize what we have defined and proved so far. For a metric space M, we considered the following four concepts: (1) compact; (2) limit point compact; (3) sequentially compact; (4) closed and bounded, and proved (1) $\to$ (4) and (2) $\to$ (3). We also saw by examples that (4) $\not\to$ (3). Unfortunately, however, we are still far away from the finish of this section, and what remain are the following:

(a) (1) $\to$ (2).

(b) (3) $\to$ (4).

(c) The equivalence between (1) and (3). This is true for any metric space.

(d) Heine-Borel theorem: the equivalence between (1) and (4) if $M = \mathbb{R}^n$. We saw that this is not true for a general metric space.

For the remaining of this section, we will prove the first two statements and the Heine-Borel theorem. Statement (c) needs some enormous extra efforts to prove, so we will omit it. Those who are interested in it may consult other books such as Marsden, or wait until the topology class.

We next state the following theorem.

Theorem 1. *Every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

Proof. We leave the proof of this theorem as a homework problem. □

Now we prove (b) sequential compactness implies 'closed and bounded'ness.

Theorem 2. *If* $A \subset M$ *is sequentially compact, then it is closed and bounded.*

Proof. Suppose A is not bounded. Then for each $n \in \mathbb{N}$, one can find a sequence $x_n \in A$ such that $d(x_1, x_n) \geq n-1$. This means that any subsequence of x_n is not bounded, hence not convergent. This contradicts the assumption that A is sequentially compact, since $x_n \in A$. We conclude that A is bounded. Next, suppose that A is not closed. Then there exists $x \in A' \setminus A$. But $x \in A'$ implies the existence of a sequence $x_n \in A$ that converges to x. Then any subsequence of x_n also converges to x, hence x_n has no subsequence that converges to a point in A. This is also a contradiction, and we conclude that A must be closed. □

As we saw before, the converse of the above theorem is not true in a general metric space. However, we have a different story in $\mathbb{R}^n$.

Theorem 3 (Bolzano-Weierstrass)**.** *A set $A \subset \mathbb{R}^n$ is closed and bounded if and only if A is sequentially compact.*

Proof. Note that we only need to prove 'closed and bounded'ness implies sequential compactness. The other direction is proved in Theorem 2.
Suppose $A \subset \mathbb{R}^n$ is closed and bounded, and x_n is a sequence in A. Then x_n is a bounded sequence in $\mathbb{R}^n$, so x_n has a convergent subsequence by Theorem 1. Suppose x_{n_j} is a convergent subsequence and $x \in \mathbb{R}^n$ is the limit of x_{n_j}. Then x must belong to $\overline{A}$ since x_{n_j} is a sequence in A. Thus $x \in A$ since A is closed. Note that we have shown that x_n has a convergent subsequence whose limit is in A, hence we conclude that A is sequentially compact since x_n is an arbitrary sequence in A. The proof has been completed. □

Next we prove that (a) compactness implies limit point compactness.

Theorem 4. *If $A \subset M$ is compact, then A is limit point compact.*

Proof. Suppose A is not limit point compact. Then there exists an infinite subset $B \subset A$ such that $B' \cap A = \emptyset$. Thus for each $a \in A$, there exists $\epsilon_a > 0$ such that $N'(a, \epsilon_a) \cap B = \emptyset$, since otherwise a would be a point in $B' \cap A$. For each $a \in A$, let $U_a = N(a, \epsilon_a)$, and note that the collection $\{U_a : a \in A\}$ covers A. But for each $a \in A$, $U_a \cap B = \{a\}$ if $a \in B$, and $U_a \cap B = \emptyset$ if $a \in A \setminus B$. Since $B \subset A$, this means that any subcover of $\{U_a : a \in A\}$ contains the collection $\{U_a : a \in B\}$. But B is an infinite set, and we conclude that the cover $\{U_a : a \in A\}$ has no finite subcover. This proves that A is not compact, so the theorem has been proved. □

For the Heine-Borel theorem, we need a series of lemmas.

Lemma 5. *A closed and bounded interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $J_0 = [a, b]$, and suppose that J_0 is not compact. Then there exists an open cover $\mathcal{C} := \{U_\alpha : \alpha \in I\}$ of J_0 which does not have a finite subcover. Let us consider the two subintervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$. Definitely $\mathcal{C}$ covers both intervals, hence one of them would not be covered by a finite sub-collection of $\mathcal{C}$. (Otherwise J_0 would be covered by a finite sub-collection

of $\mathcal{C}$.) Let J_1 be the one which cannot be covered by a finite sub-collection of $\mathcal{C}$. If both subintervals have this property, we can choose either one.
We repeat this process, and get a sequence of closed and bounded intervals $J_0 \supset J_1 \supset J_2 \supset \cdots$ such that the length of J_n is $(b-a)/2^n$, and J_n cannot be covered by any finite sub-collection of $\mathcal{C}$. Now by the nested intervals theorem, there exists $x \in [a,b]$ such that $\{x\} = \bigcap_{n=1}^{\infty} J_n$.
Since $x \in J_0$ and J_0 is covered by $\mathcal{C}$, there exists an open set $U_{\alpha_0} \in \mathcal{C}$ such that $x \in U_{\alpha_0}$. But because U_{α_0} is open, there exists $\epsilon > 0$ such that $N(x,\epsilon) = (x-\epsilon, x+\epsilon) \subset U_{\alpha_0}$. Hence if we take n sufficiently large so that $(b-a)/2^n < \epsilon$, we have $J_n \subset U_{\alpha_0}$. This is a contradiction, however, since $\{U_{\alpha_0}\}$ is a finite sub-collection of $\mathcal{C}$ which covers J_n, while we chose J_n so that any sub-collection of $\mathcal{C}$ cannot cover J_n. This contradiction proves the theorem. □

One can actually repeat the above process, and should be able to prove the following lemma.

Lemma 6. *The set*

$$[-L,L]^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |x_j| \le L \text{ for all } j = 1, \ldots, n.\}$$

is compact.

The next lemma is also important by itself.

Lemma 7. *Any closed subset of a compact set is compact.*

Proof. Suppose $A \subset M$ is compact and $B \subset A$ is closed. Let $\mathcal{C} = \{U_\alpha : \alpha \in I\}$ be an open cover of B. Since B is closed, the set $U := M \setminus B$ is open, hence the collection $\mathcal{C} \cup \{U\}$ is an open cover of A. Since A is compact, there exists a finite sub-collection $\mathcal{C}_0 \subset \mathcal{C} \cup \{U\}$ which covers A. Then one can easily see that the collection $\mathcal{C}_0 \setminus \{U\}$ is a finite sub-collection of $\mathcal{C}$ that covers B, hence a finite subcover of $\mathcal{C}$. Since $\mathcal{C}$ is an arbitrary open cover of B, this shows that B is compact. □

Now we are ready to prove the Heine-Borel Theorem.

Theorem 8 (Heine-Borel)**.** *$K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.*

Proof. Suppose K is closed and bounded subset in $\mathbb{R}^n$. Then since K is bounded, there exists $L > 0$ such that $K \subset [-L, L]^n$. But $[-L, L]^n$ is compact by Lemma 6, so K is a closed set in a compact set. Now Lemma 7 implies that K is compact. The other direction of the statement has been proved before, so the theorem follows. □