Fourier Series

- 주기 [전력] 신호인 경우, 기본 주기 T_0
- 정현 Fourier Series
  - 직교 함수 집합 : 구간 $t_0 \le t \le t_0 + T_0$

$$\{\phi_n(t)\} = \{1,\ \cos n\omega_o t,\ \sin n\omega_o t,\ n = 1, 2, 3, \cdots, \infty\}$$

- 복소 지수함수 Fourier Series
  - 직교 함수 집합 : 구간 $t_0 \le t \le t_0 + T_0$

$$\{\phi_n(t)\} = \{e^{jn\omega_o t},\ n = 0,\ \pm 1,\ \pm 2,\ \cdots,\ \pm\infty\} \quad \left(\omega_o = \frac{2\pi}{T_0}\right)$$

복소 지수함수 Fourier Series

- 복소 지수함수 근저 함수 (basis function) 집합에 의한 표현
  - 구간 $t_0 \le t \le t_0 + T_0$ 에서 임의의 신호 $x(t)$는 아래 선형 조합으로 표현

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{jn\omega_o t}, \quad \omega_o = \frac{2\pi}{T_0}$$

  - 선형조합의 계수 C_n은 다음과 같이 구함

$$C_n = \frac{< x(t), \phi_n(t) >}{\int_{t_o}^{t_o+T_0} |\phi_n(t)|^2 \, dt} = \frac{\int_{t_o}^{t_o+T_0} x(t)\phi_n^*(t)dt}{\int_{t_o}^{t_o+T_0} 1 \, dt} = \frac{1}{T_0}\int_{t_o}^{t_o+T_0} x(t)e^{-jn\omega_o t}dt$$

  - 만일 신호 $x(t)$가 주기 T_0를 가진 주기 신호라면 basis 함수들도 같은 주기의 주기 함수가 되므로 위의 무한급수 표현은 한 주기의 구간을 넘어서 모든 시구간 $-\infty < t < \infty$ 로 확장할 수 있음

(3) The inner product in time :

Let's define : $< x(t),\, e^{jk\omega_0 t} > \; \equiv \dfrac{1}{T}\displaystyle\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \;= X(k)$

then : $< e^{jm\omega_0 t},\, e^{jk\omega_0 t} > \; = \dfrac{1}{T}\displaystyle\int_0^T e^{jm\omega_0 t}\, e^{-jk\omega_0 t}\, dt = \dfrac{1}{T}\displaystyle\int_0^T e^{j(m-k)\omega_0 t}\, dt$

$$[\because \; \omega_0 T = 2\pi] \quad = \begin{cases} \dfrac{1}{T}\displaystyle\int_0^T 1\, dt = 1, & m = k \\ \dfrac{1}{T}\left[\dfrac{e^{j(m-k)\omega_0 t}}{j(m-k)\omega_0}\right]_0^T = 0, & m \neq k \end{cases}$$

$$= \; \delta(m-k)$$

- Fourier Series 및 Fourier Series 계수

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0} \qquad \text{(Fourier Series: Synthesis)}$$

$$C_n = \frac{1}{T_0}\int_{T_o} x(t)e^{-jn\omega_o t}dt = \frac{1}{T_0}\int_{T_o} x(t)e^{-j2\pi nt/T_0}dt \qquad \text{(FS Coefficients: Analysis)}$$

- FA Coefficient C_n은 $n\omega_0$ 주파수 성분(nth harmonic)의 전력량
- 일명, 스펙트럼 계수(spectrum coefficient)
- 복소수 : $C_n = |C_n| e^{j \arg C_n}$, 여기서 $\arg C_n$은 C_n의 편각(argument)

Fourier Series 성질

❑ 대칭성

- $x(t)$의 값이 real인 경우, 즉 $x(t) = x^*(t)$인 경우

  $c_n = c^*_{-n}$ (conjugate symmetric)

  $\Rightarrow |c_{-n}| = |c_n|$ (amplitude spectrum is even)

  $\arg c_{-n} = -\arg c_n$ (phase spectrum is odd)

- If $x(t)$ is real & even (i.e. $x(t) = x(-t)$), then c_n is also real & even

- If $x(t)$ is real & odd (i.e. $x(t) = -x(-t)$), then c_n is imaginary & odd

❑ 선형성

- $x(t)$와 $y(t)$가 동일한 주기를 갖는 주기신호인 경우

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}$$

$$y(t) = \sum_{n=-\infty}^{\infty} d_n \, e^{jn\omega_0 t}$$

- $z(t) = \alpha x(t) + \beta y(t)$ 의 푸리에 급수는

$$z(t) = \sum_{n=-\infty}^{\infty} g_n \, e^{jn\omega_0 t}$$

$$\text{with } \; g_n = \alpha \, c_n + \beta \, d_n$$

❑ 시간 천이

- $x(t)$의 푸리에 계수가 c_n 인 경우
- $x(t\text{-}\tau)$의 푸리에 계수 d_n 은

$$d_n = \frac{1}{T_0}\int_{T_o} x(t-\tau)e^{-jn\omega_o t}dt = \frac{1}{T_0}\int_{T_o} x(t-\tau)e^{-jn\omega_o (t-\tau)}e^{-jn\omega_0 \tau}dt$$

$$= e^{-jn\omega_0 \tau}\frac{1}{T_0}\int_{T_o} x(\lambda)e^{-jn\omega_o \lambda}d\lambda$$

$$= e^{-jn\omega_0 \tau}\cdot c_n$$

- 진폭 스펙트럼은 변화가 없다.

$$|d_n| = |c_n|$$

Parseval's Theorem

- 주기 신호의 평균 전력

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

- Real valued signal의 경우

$$P = |c_0|^2 + 2\sum_{n=1}^{\infty} |c_n|^2$$

- $|c_n|^2$ is called power spectrum

- [proof]
  - $z(t) = x(t)y^*(t)$의 푸리에 계수

    $$z(t) = x(t)y^*(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t} \sum_{m=-\infty}^{\infty} d_m^*\, e^{-jm\omega_0 t}$$

    $$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n\, d_m^*\, e^{j(n-m)\omega_0 t} = \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} c_{k+m}\, d_m^* \right) e^{jk\omega_0 t}$$

  - $z(t)$의 푸리에 계수

    $$\sum_{m=-\infty}^{\infty} c_{k+m}\, d_m^* = \frac{1}{T_0} \int_{T_0} x(t)y^*(t)e^{-jk\omega_0 t}\, dt$$

  - $z(t)$의 직류값($k = 0$에서의 푸리에 계수)

    $$\sum_{m=-\infty}^{\infty} c_m\, d_m^* = \frac{1}{T_0} \int_{T_0} x(t)y^*(t)\, dt$$

  - $x(t) = y(t)$인 경우

    $$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T_0} \int_{T_0} |x(t)|^2\, dt$$

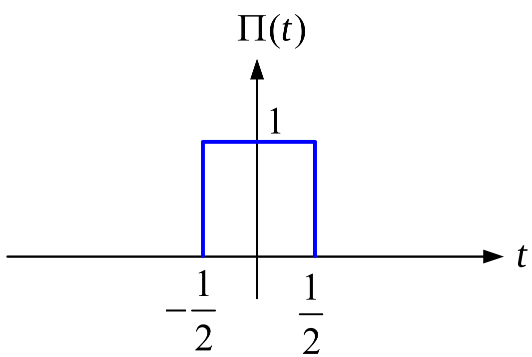
Rectangular Pulse

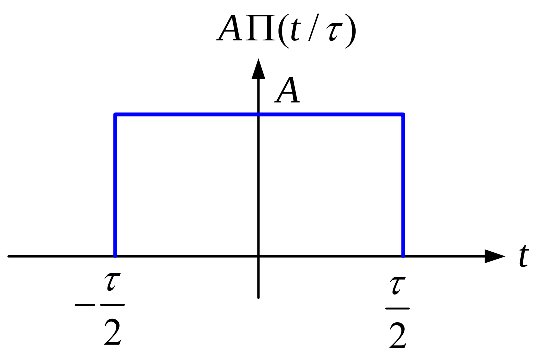
- 크기가 1이고 펄스 폭이 1인 사각 펄스(구형파)

$$\Pi(t)[=rect(t)]=\begin{cases}1 & \text{for } -\frac{1}{2}<t<\frac{1}{2}\\ 0 & \text{otherwise}\end{cases}$$

$$=u\left(t+\frac{1}{2}\right)-u\left(t-\frac{1}{2}\right)$$

- 크기가 A이고 펄스 폭이 τ인 사각 펄스: $A\Pi(t/\tau)$



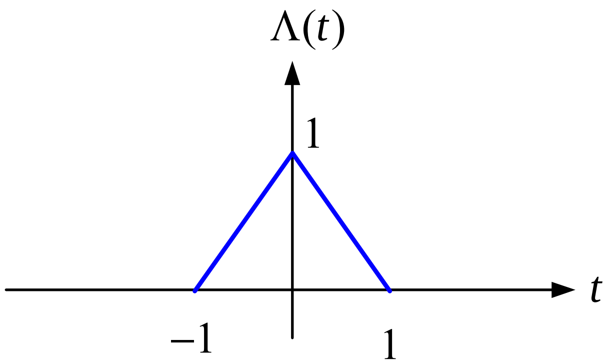


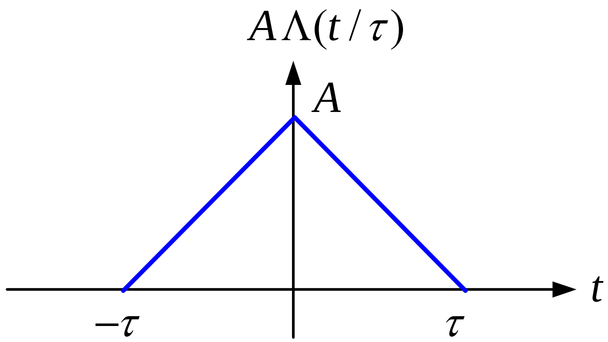
Triangular Pulse

- 크기가 1이고 펄스 폭이 2인 삼각 펄스

$$\Lambda(t) = \begin{cases} 1-|t| & \text{for } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- 크기가 A이고 펄스 폭이 2τ인 삼각 펄스: $A\Lambda(t/\tau)$



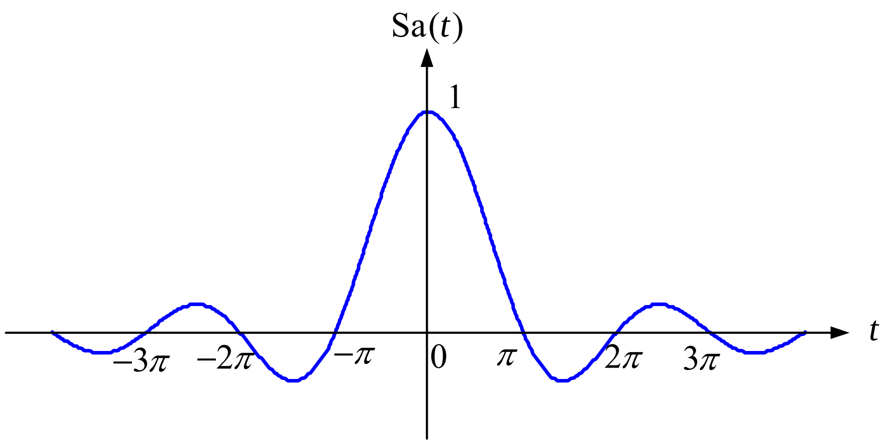


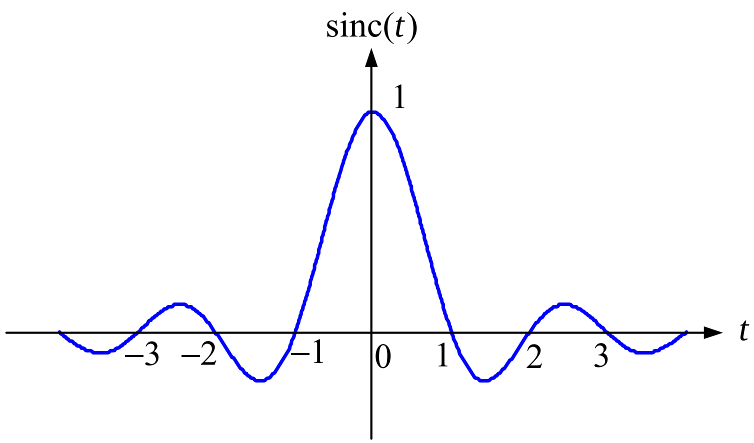
Sampling Function

- Sa 함수와 sinc 함수

$$\mathrm{Sa}(t) = \frac{\sin t}{t}$$

$$\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t} = \mathrm{Sa}(\pi t)$$

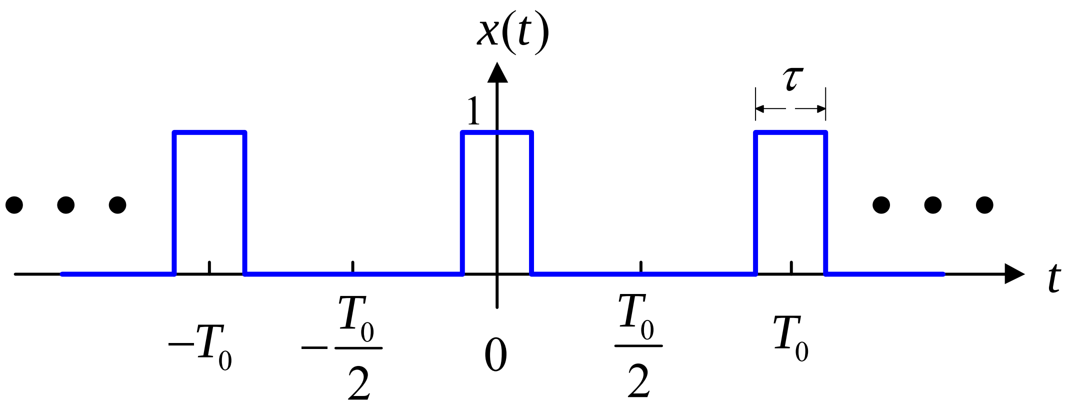




❑ [Example] Rectangular Pulse Train

- 주기가 T_0이고 펄스폭이 τ인 구형 펄스 열

$$x(t) = \sum_{k=-\infty}^{\infty} \Pi\left(\frac{t-kT_0}{\tau}\right)$$



❑ [Example] Rectangular Pulse Train

$$\omega_0 = 2\pi / T_0 = \pi / 2$$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_o t}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\exp\left(\frac{-j2\pi nt}{T_0}\right)dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} (1)\exp\left(\frac{-j2\pi nt}{T_0}\right)dt$$

i) $n = 0$: dc value

$$c_0 = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} (1)dt = \frac{\tau}{T_0}$$

ii) $n \neq 0$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\exp\left(\frac{-j2\pi nt}{T_0}\right)dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2}(1)\exp\left(\frac{-j2\pi nt}{T_0}\right)dt$$

$$= \frac{1}{-j2\pi n}\exp\left(\frac{-j2\pi nt}{T_0}\right)\Bigg|_{-\tau/2}^{\tau/2}$$

$$= \frac{1}{j2\pi n}\left[\exp\left(\frac{j\pi n\tau}{T_0}\right) - \exp\left(\frac{-j\pi n\tau}{T_0}\right)\right]$$

$$= \frac{\sin\left(\pi n\tau / T_0\right)}{\pi n} = \frac{\tau}{T_0}\frac{\sin\left(\pi n\tau / T_0\right)}{\left(\pi n\tau / T_0\right)}$$

$$= \frac{\tau}{T_0}\operatorname{sinc}\left(\frac{n\tau}{T_0}\right) = \frac{\tau}{T_0}\operatorname{sinc}\left(nf_0\tau\right)$$

$$= \frac{\tau}{T_0}Sa\left(\frac{\pi n\tau}{T_0}\right) = \frac{\tau}{T_0}Sa\left(\frac{n\omega_0\tau}{2}\right)$$

❑ [Example] Rectangular Pulse Train

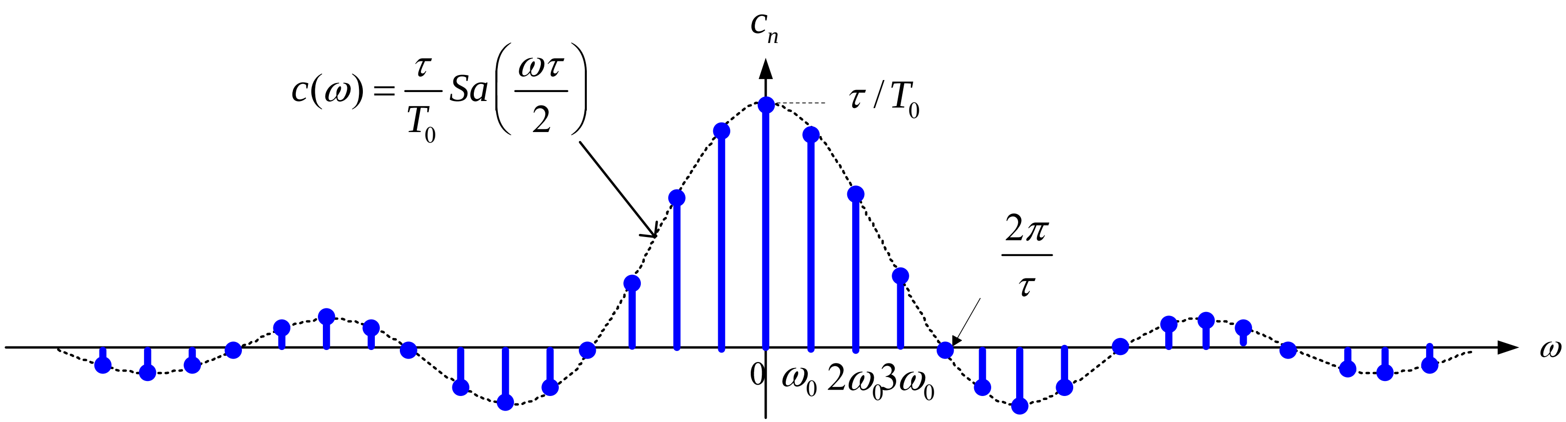
- Spectrum shape :

$$c_n = c(n\omega_0) = \frac{\tau}{T_0} Sa\left(\frac{n\omega_0\tau}{2}\right) = \frac{\tau}{T_0} Sa\left(\frac{\omega\tau}{2}\right)\Bigg|_{n\omega_0}$$

$$c(\omega) = \frac{\tau}{T_0} Sa\left(\frac{\omega\tau}{2}\right)$$

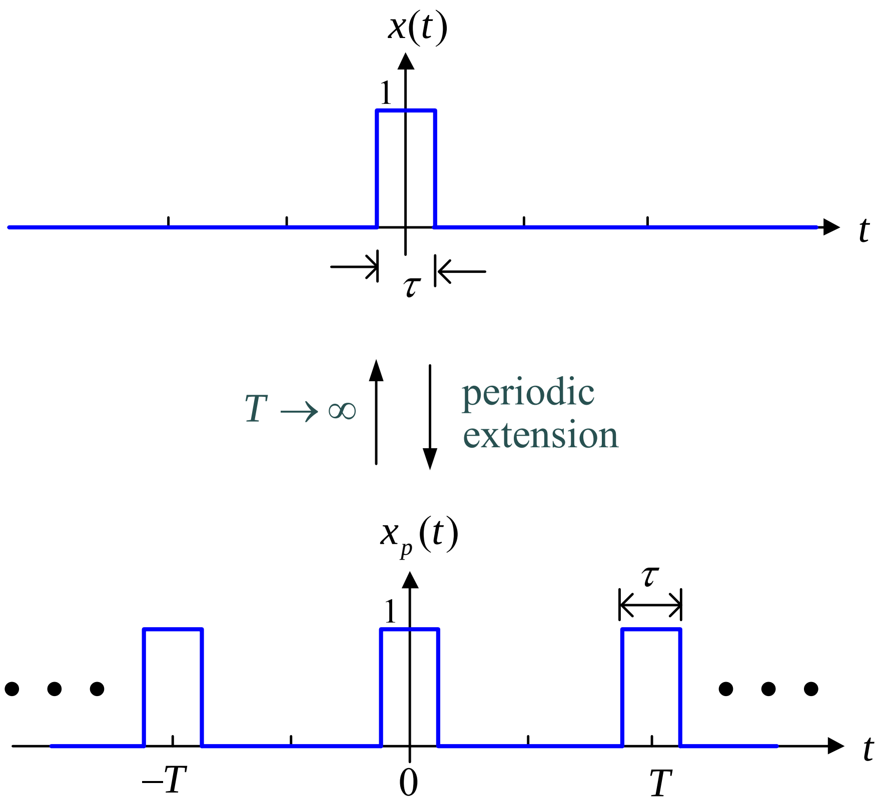
- $c(\omega)$ 의 파형은 진동하면서 감쇠하고, $\omega = n\pi / \tau \ (n \neq 0)$마다 zero-crossing

- First zero-crossing frequency: $\omega = \pi / \tau$



비주기 신호의 Fourier Transform

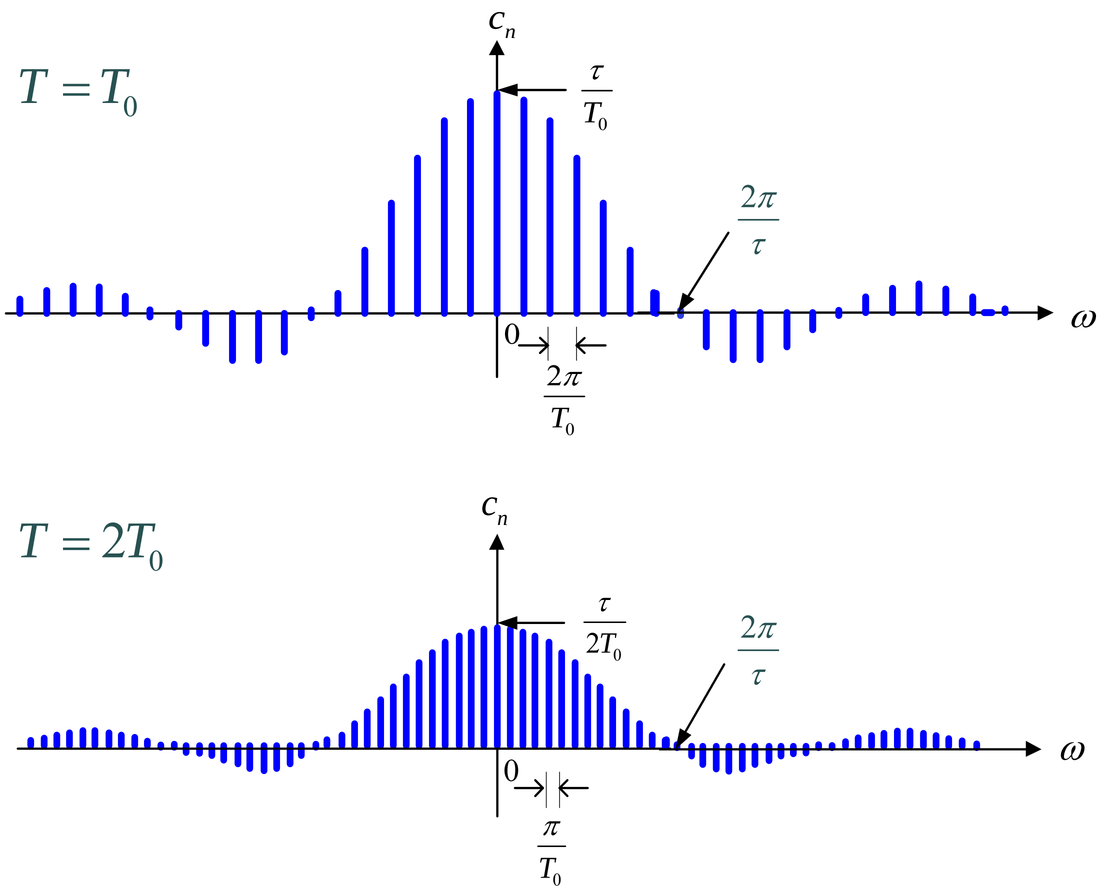
❑ 비주기 신호의 주기적 확장



$$x_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t) e^{-jn\omega_o t} dt \quad (\omega_0 = 2\pi / T)$$

$$c_n = \frac{\tau}{T} Sa\left(\frac{\omega\tau}{2}\right)\Bigg|_{\omega = n\omega_0} = \frac{\tau}{T} Sa\left(\frac{n\omega_0\tau}{2}\right)$$



1. Spectrum shape :

$$c(\omega) = \frac{\tau}{T} Sa\left(\frac{\omega\tau}{2}\right)$$

2. First zero-crossing : $\frac{\omega\tau}{2} = \pi$

$$\omega = n\omega_0 = n\frac{2\pi}{T} = \frac{2\pi}{\tau}$$

- Define discrete frequency ω_n as

$$\omega_n \triangleq n\omega_0 = n\frac{2\pi}{T}$$

- Define $X(\omega_n)$

$$X(\omega_n) \triangleq Tc_n = \int_{-T/2}^{T/2} x_p(t)\, e^{-jn\omega_o t} dt$$

- Let $T \to \infty$ then

$$X(\omega_n) = Tc_n \xrightarrow{T\to\infty} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \triangleq X(\omega)$$

$$x(t) = \lim_{T\to\infty} x_p(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_o t} = \lim_{T\to\infty} \frac{1}{T} \sum_{n=-\infty}^{\infty} Tc_n\, e^{jn\omega_o t}$$

$$= \lim_{\substack{T\to\infty \\ \omega_0 \to 0}} \sum_{n=-\infty}^{\infty} [Tc_n] e^{jn\omega_o t} \cdot \frac{\omega_0}{2\pi} \quad = \int_{-\infty}^{\infty} [X(\omega)] e^{j\omega t} \frac{d\omega}{2\pi}$$

- Fourier Transform

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

- Inverse Fourier Transform

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} \frac{d\omega}{2\pi}$$

- 주파수 변수를 ω 대신 f를 사용하면

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df$$

❑ Continuous Spectrum

- 주기 신호의 푸리에 스펙트럼이 이산 스펙트럼
- 비주기 신호의 푸리에 스펙트럼은 연속 스펙트럼

❑ Amplitude spectrum and phase spectrum

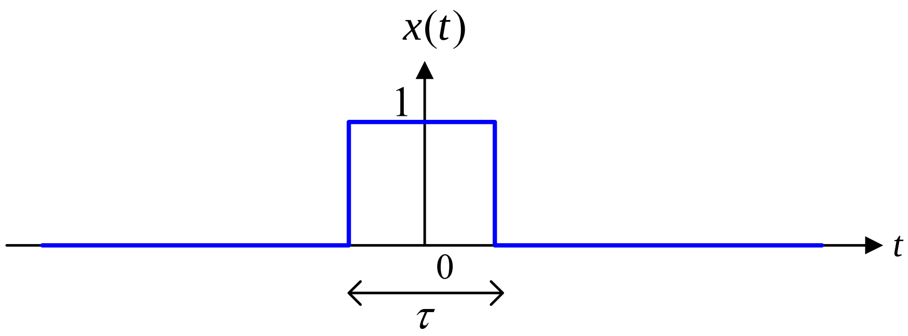
$$X(\omega) = \left|X(\omega)\right| e^{j \arg X(\omega)}$$

1) $\left|X(\omega)\right|$: amplitude spectrum

2) $\arg X(\omega)$: phase spectrum

❑ [예제 5.1] Rectangular Pulse with pulse width τ

$$x(t) = \Pi(t/\tau) = \begin{cases} 1 & |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$



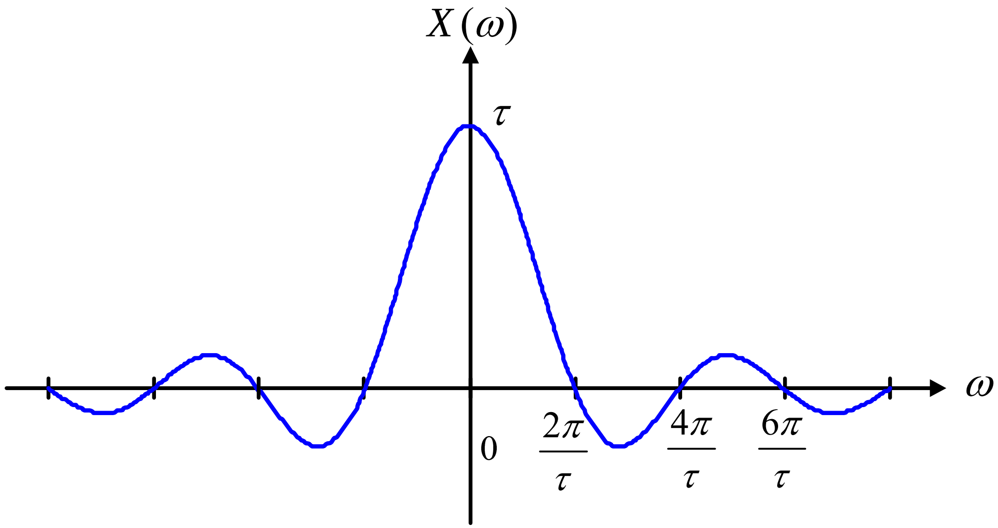
$$\mathcal{F}\{\Pi(t/\tau)\} = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt$$

$$= \begin{cases} If\ \ \omega = 0, & \tau \\ If\ \ \omega \ne 0, & \left[\dfrac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2} = \tau \dfrac{\sin\left(\dfrac{\omega\tau}{2}\right)}{\left(\dfrac{\omega\tau}{2}\right)} \end{cases}$$

$$= \tau\, Sa\left(\frac{\omega\tau}{2}\right)$$

❑ [예제 5.1] Rectangular Pulse with pulse width τ

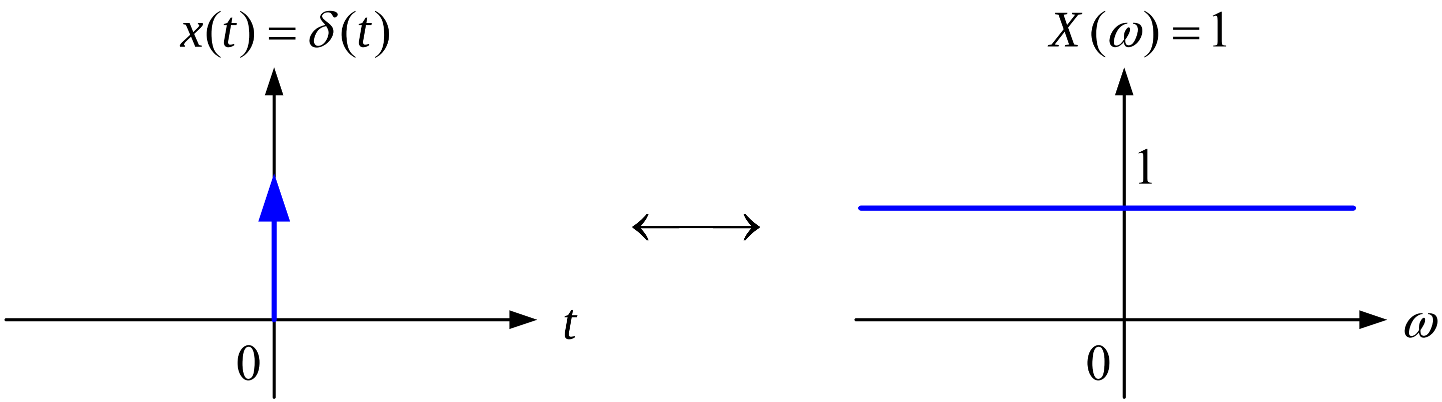
$$\Pi(t/\tau) \stackrel{F}{\longleftrightarrow} \tau\, Sa\left(\frac{\omega\tau}{2}\right)$$



❑ [예제 5.2] Impulse

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{0} \int_{-\infty}^{\infty} \delta(t) dt = 1$$

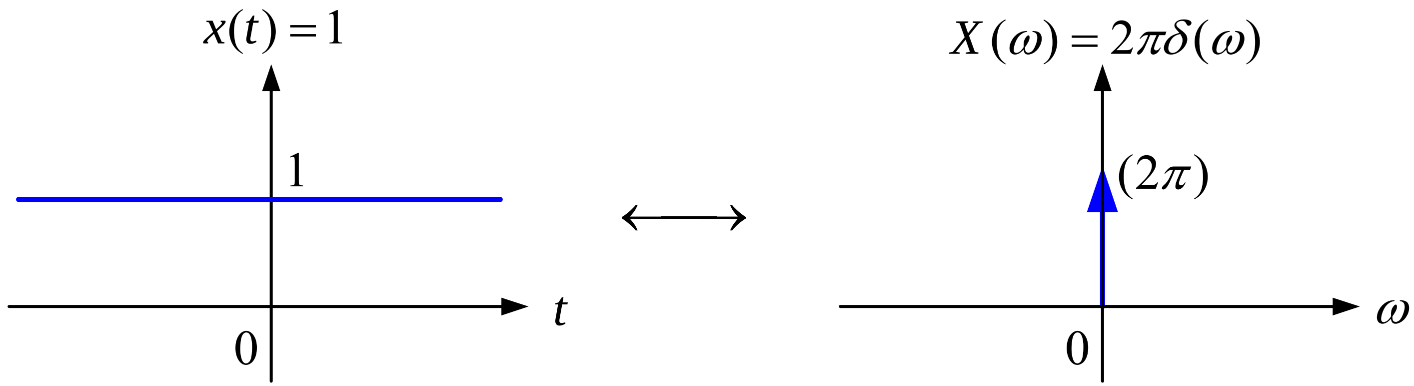
$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$



❑ [예제 5.3] Inverse Fourier transform of $\delta(\omega)$

$$\mathcal{F}^{-1}\{\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \delta(\omega) e^{0} \frac{d\omega}{2\pi} = \frac{1}{2\pi}$$

$$1 \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\omega) = \delta(f)$$



❑ [예제 5.4] Inverse Fourier transform of rectangular spectrum

- Rectangular spectrum of bandwidth $\tau / 2$

$$X(\omega) = \Pi(\omega / \tau) = \begin{cases} 1 & |\omega| < \tau / 2 \\ 0 & |\omega| > \tau / 2 \end{cases}$$

$$x(t) = \mathcal{F}^{-1}\{\Pi(\omega/\tau)\} = \int_{-\tau/2}^{\tau/2} e^{j\omega t} \frac{d\omega}{2\pi}$$

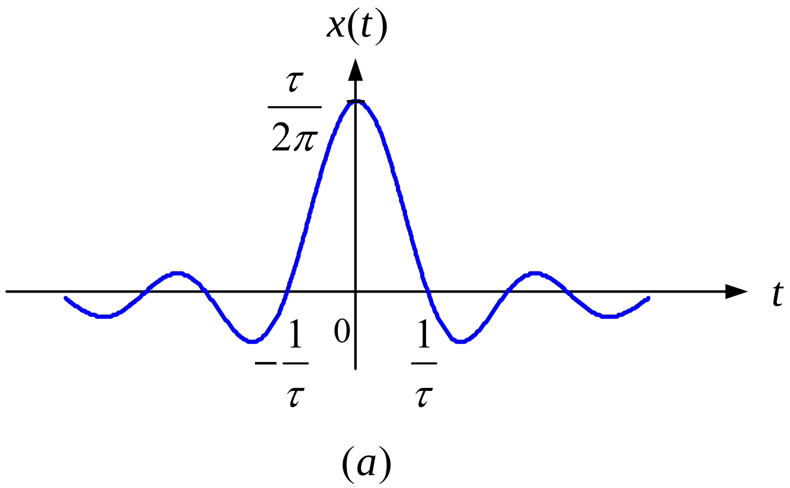
$$= \begin{cases} If\ t = 0, & \dfrac{\tau}{2\pi} \\ If\ t \neq 0, & \left[\dfrac{1}{2\pi}\dfrac{\exp(j\omega t)}{jt}\right]_{-\tau/2}^{\tau/2} = \dfrac{\tau}{2\pi}\dfrac{\sin\left(\dfrac{\tau t}{2}\right)}{\left(\dfrac{\tau t}{2}\right)} \end{cases}$$

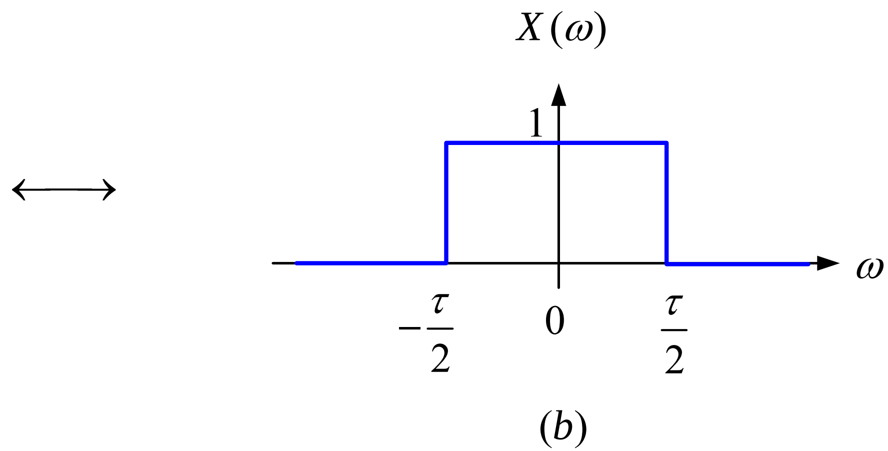
$$= \frac{\tau}{2\pi} Sa\left(\frac{\tau t}{2}\right)$$

❑ [예제 5.4] Inverse Fourier transform of rectangular spectrum

- Rectangular spectrum of bandwidth $\tau / 2$

$$\frac{1}{2\pi} \cdot \tau\, Sa\left(\frac{\tau t}{2}\right) \xleftrightarrow{F} \Pi\left(\frac{\omega}{\tau}\right)$$





❑ [예제 5.5] Exponential function

$$x(t) = e^{-at}u(t),\ a > 0$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}u(t)dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{a + j\omega}, \quad a > 0$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} = \frac{1}{a} \cdot \frac{1}{\sqrt{1 + \left(\frac{\omega}{a}\right)^2}}$$

$$|X(a)| = |X(0)| \cdot \frac{1}{\sqrt{2}}$$

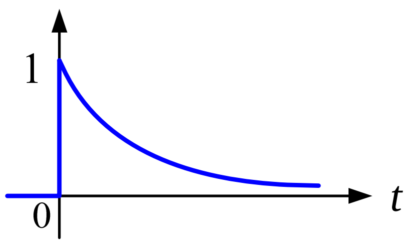
$$\arg X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\arg X(a) = -\frac{\pi}{4}$$

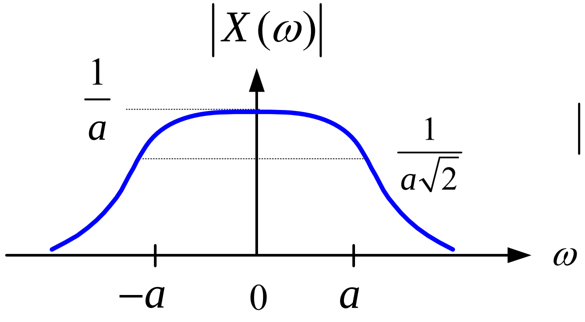
❑ [예제 5.5] Exponential function

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

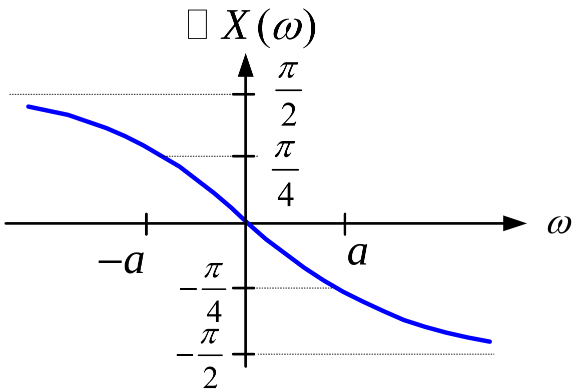
$x(t)=e^{-at}u(t)$



$\longleftrightarrow$



$$|X(\omega)| = \frac{1}{a}\frac{1}{\sqrt{1+\left(\frac{\omega}{a}\right)^2}}$$



$$\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$