Chapter 7. Electrodynamics

James Clerk Maxwell

a Scottish mathematical physicist
1831-1879
http://en.wikipedia.org/wiki/James_Clerk_Maxwell

7.1 Electromotive Force

7.2 Electromagnetic Induction

7.3 Maxwell's Equations

Correction on the **Ampere's Law :** $\nabla\times\vec{B}=\mu_0\vec{J}$ → $\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

Maxwell's Law : $\nabla\cdot\vec{E}=\frac{1}{\varepsilon_0}\rho$ $\quad\nabla\cdot\vec{B}=0$ $\quad\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ $\quad\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

The **Lorentz force law :** $\vec{F}=q\left(\vec{E}+\vec{v}\times\vec{B}\right)$

The **Continuity equation :** $\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}$ (Conservation law of charge)

Maxwell equations in matter with **electric** and **magnetic polarization** (ρ_b, J_b),

$\nabla\cdot\vec{D}=\rho_f,$ $\quad\nabla\cdot\vec{B}=0,$ $\quad\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ $\quad\nabla\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}$

Boundary conditions for electrodynamics :

$D_1^{\perp}-D_2^{\perp}=\sigma_f$ $\quad E_1^{//}-E_2^{//}=0$ $\quad\vec{H}_1^{//}-\vec{H}_2^{//}=\vec{K}_f\times\hat{n}$ $\quad B_1^{\perp}-B_2^{\perp}=0$

7.3 Maxwell's Equations

7.3.1 Electrodynamics Before Maxwell

- **Divergence** and **Curl** of **Electric** and **Magnetic Fields**

$\int_V (\nabla \cdot \vec{E}) dV = \oint_S \vec{E} \cdot d\vec{a}$

: Divergence Theorem

(i) $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$: Gauss's law

(ii) $\nabla \cdot \vec{B} = 0$: No name (No magnetic monopoles)

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: Faraday's law

(iv) $\nabla \times \vec{B} = \mu_0 \vec{J}$: Ampere's law

This is the electromagnetic theory before the mid-nineteenth century (before Maxwell's work)

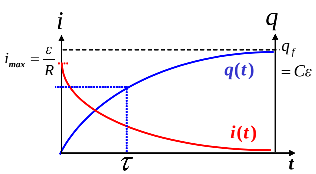
If the divergence is applied to the third equation, $\nabla \cdot (\nabla \times \vec{E}) = \nabla \cdot \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \cdot \vec{B}) = 0$ → Correct!

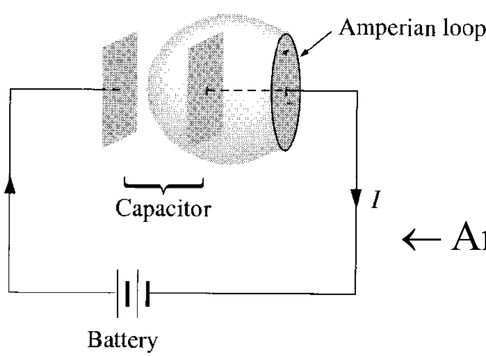
If the divergence is applied to the third equation, $\underbrace{\nabla \cdot (\nabla \times \vec{B})}_{=0} = \mu_0 \underbrace{(\nabla \cdot \vec{J})}$ → Not correct!

$= 0$ for steady currents

$\neq 0$ for non-steady currents

Ampere's law is not correct beyond magnetostatics





← An exceptional case to Ampere's law: **Nonsteady currents**

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc.}$$

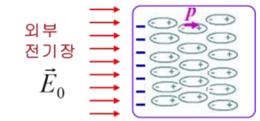
$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \oint_{Loop} \vec{B} \cdot d\vec{l}$

: Curl Theorem

7.3 Maxwell's Equations

7.3.2 How Maxwell Fixed Ampere's Law

$$\underbrace{\nabla\cdot(\nabla\times\vec{B})}_{=0}=\mu_0\underbrace{(\nabla\cdot\vec{J})}_{\substack{=0 \text{ for steady currents}\\ \neq 0 \text{ for non-steady currents}}}$$



$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}=\varepsilon_0(1+\chi_e)\vec{E}=\varepsilon\,\vec{E}$$

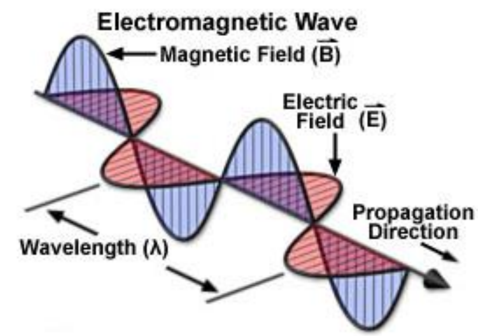
$$\vec{B}=\mu_0(\vec{H}+\vec{M})=\mu_0(1+\chi_m)\vec{H}=\mu\,\vec{H}$$

$$\nabla\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}$$

Eq. (5.29) → The continuity equation

$$\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot\vec{E})=-\nabla\cdot\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow\quad \nabla\cdot\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)=0$$



Thus, Ampere's law can be corrected as

$$\nabla\times\vec{B}=\mu_0\vec{J}+\underbrace{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}_{=\mu_0\vec{J}_d} \qquad (7.37)$$

: a **correct Ampere's law**

Due to conductor ($\mu_0\vec{J}$)

Due to dielectrics ($\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$)

➡ This term plays a crucial role in the propagation of electromagnetic waves.

$$\vec{J}_d\equiv\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad (7.38)$$

: the displacement current (? no good name)

⇒ A changing electric field induces a magnetic field.

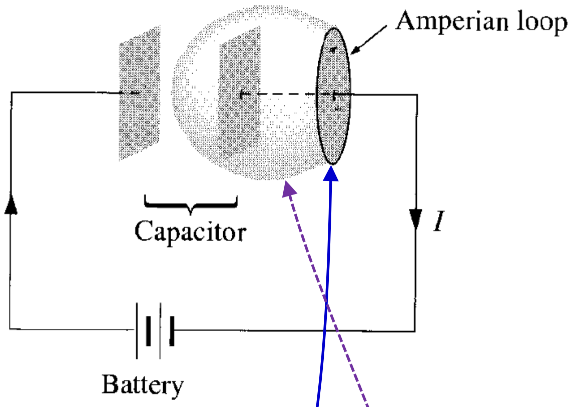
7.3 Maxwell's Equations

7.3.2 How Maxwell Fixed Ampere's Law - *continued*

If the capacitor plates are very close together, the electric field between them is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{A}$$

where Q is the charge on the plate and A is its area.



Then, $$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt} = \frac{1}{\varepsilon_0 A} I$$

Eq. (7.37) → $$\int \left(\nabla \times \vec{B}\right)\cdot d\vec{a} = \int \left(\mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc.} + \mu_0 \varepsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a} \qquad (7.39)$$

For the flat Amperian loop, $$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc.}$$

For the balloon-shaped Amperian loop, $$\oint \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a} = \mu_0 \varepsilon_0 \frac{1}{\varepsilon_0 A} I \cdot A = \mu_0 I$$

7.3 Maxwell's Equations

7.3.3 Maxwell's Equations

(i) $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho$: Gauss's law

(ii) $\nabla \cdot \vec{B} = 0$: No name

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$: Faraday's law

(iv) $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$: Ampere's law with Maxwell's correction

(7.40)

$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$: Lorentz force law (7.41)

$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$: Continuity equation (Conservation law of charge) (7.42)

Full theory of **classical electrodynamics**

Maxwell's Eq. (iv) → $\nabla \cdot \left(\nabla \times \vec{B}\right) = \mu_0 \nabla \cdot \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(\nabla \cdot \vec{E}\right) = 0$

$$\nabla \cdot \vec{J} + \varepsilon_0 \frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right) = 0 \quad \Rightarrow \quad \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

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7.3 Maxwell's Equations

Maxwell's Equations

General form:

$$\nabla\cdot\vec{E}=\frac{1}{\varepsilon_0}\rho, \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B}=0, \qquad \nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

In free space ($\rho = 0$, $J = 0$),

$$\nabla\cdot\vec{E}=0, \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B}=0, \qquad \nabla\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

In Matter with **electric** and **magnetic polarization** (ρ_b, J_b),

$$\nabla\cdot\vec{D}=\rho_f, \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B}=0, \qquad \nabla\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t} \qquad (7.56)$$

$\vec{D}$: Electric displacement

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}=\varepsilon_0(1+\chi_e)\vec{E}=\varepsilon\vec{E}$$

$$\varepsilon=\varepsilon_0(1+\chi_e)=\varepsilon_0\varepsilon_r$$

$$\vec{B}=\mu_0(\vec{H}+\vec{M})=\mu_0(1+\chi_m)\vec{H}=\mu\vec{H}$$

$$\mu=\mu_0(1+\chi_m)$$

$$\vec{H}=\frac{\vec{B}}{\mu} \qquad (7.58)$$

$$\vec{J}_d\equiv\frac{\partial\vec{D}}{\partial t} \qquad (7.59)$$

: displacement current

7.3.6 Boundary Conditions

The discontinuities of the fields at a boundary between two different media can be deduced from Maxwell's equations (7.56):

$$\int_v \left(\nabla \cdot \vec{D}\right) dv = \int_v \rho_f \, dv,$$

$$\int_v \left(\nabla \cdot \vec{B}\right) dv = 0,$$

$$\int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{a} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{a}$$

$$\int_S \left(\nabla \times \vec{H}\right) \cdot d\vec{a} = \int_S \vec{J}_f \cdot d\vec{a} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{a}$$

(i) $\oint_S \vec{D} \cdot d\vec{a} = Q_{f_{enc.}}$

(ii) $\oint_S \vec{B} \cdot d\vec{a} = 0$

} over any closed surface S

(iii) $\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{a}$

(iv) $\oint_L \vec{H} \cdot d\vec{l} = I_{f_{enc.}} + \frac{d}{dt} \int_S \vec{D} \cdot d\vec{a}$

} for any surface S bounded by the closed loop L.

Applying (i) to a very thin pillbox over the boundary as shown in figure, we obtain

$$\vec{D}_1 \cdot \vec{a} - \vec{D}_2 \cdot \vec{a} = \sigma_f a$$

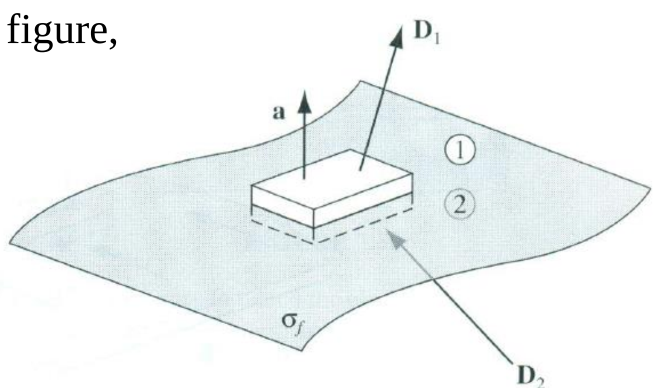
(The positive direction for ***a*** is *from* 2 toward 1.)

Thus,

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \quad (7.60)$$

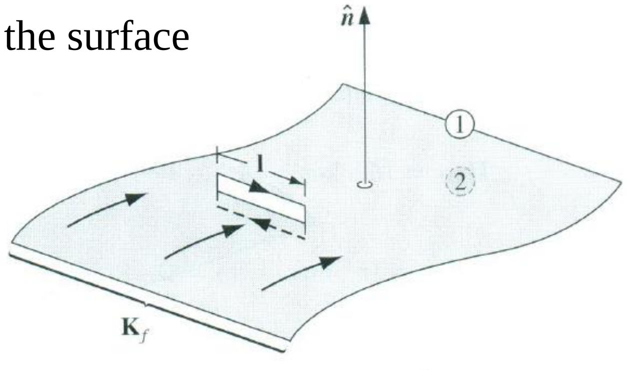
Treating (ii) in a similar way, we get

$$B_1^{\perp} - B_2^{\perp} = 0 \quad (7.61)$$



7.3.6 Boundary Conditions - *continued*

Applying the condition (iii) to a very thin Amperian loop straddling the surface as shown in figure, we obtain



$$\oint_L \vec{E}\cdot d\vec{l} = \vec{E}_1\cdot\vec{l} - \vec{E}_2\cdot\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{a}$$

= 0 (in a limit of very small area)

Thus,

$$E_1^{//} - E_2^{//} = 0 \qquad (7.62)$$

⇒ The components of ***E*** parallel to the interface are continuous across the boundary.

Managing (iv) in a similar way, we get

$$\oint_L \vec{H}\cdot d\vec{l} = \vec{H}_1\cdot\vec{l} - \vec{H}_2\cdot\vec{l} = I_{f_{enc.}} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{a} \quad \Rightarrow \quad \vec{H}_1\cdot\vec{l} - \vec{H}_2\cdot\vec{l} = I_{f_{enc.}}$$

where $I_{fenc.}$ is the free current passing through the Amperian loop.

→ For the very thin Amperian loop, no volume current density will contribute, but

→ a surface current can contribute to it.

$$I_{f_{enc.}} = \vec{K}_f\cdot\left(\hat{n}\times\vec{l}\right) = \left(\vec{K}_f\times\hat{n}\right)\cdot\vec{l}$$

($\hat{n}$ is a unit vector perpendicular to the interface pointing from 2 toward 1.)

$$\vec{H}_1^{//} - \vec{H}_2^{//} = \vec{K}_f\times\hat{n} \qquad (7.63)$$

7.3 Maxwell's Equations

7.3.6 Boundary Conditions *- continued*

Summarized general **boundary conditions for electrodynamics** :

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad E_1^{//} - E_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \vec{H}_1^{//} - \vec{H}_2^{//} = \vec{K}_f \times \hat{n}$$

In the case of **linear media** : $(\vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H})$

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \qquad E_1^{//} - E_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\vec{B}_1^{//} - \frac{1}{\mu_2}\vec{B}_2^{//} = \vec{K}_f \times \hat{n} \tag{7.64}$$

If there is **no free charge** or **free current** at the interface,

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 \qquad E_1^{//} - E_2^{//} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\vec{B}_1^{//} - \frac{1}{\mu_2}\vec{B}_2^{//} = 0 \tag{7.65}$$

Next Semester

Starting from Chapter 8. Conservation Laws

8.1 Charge and Energy

8.2 Momentum

8.3 Magnetic Forces Do No Work