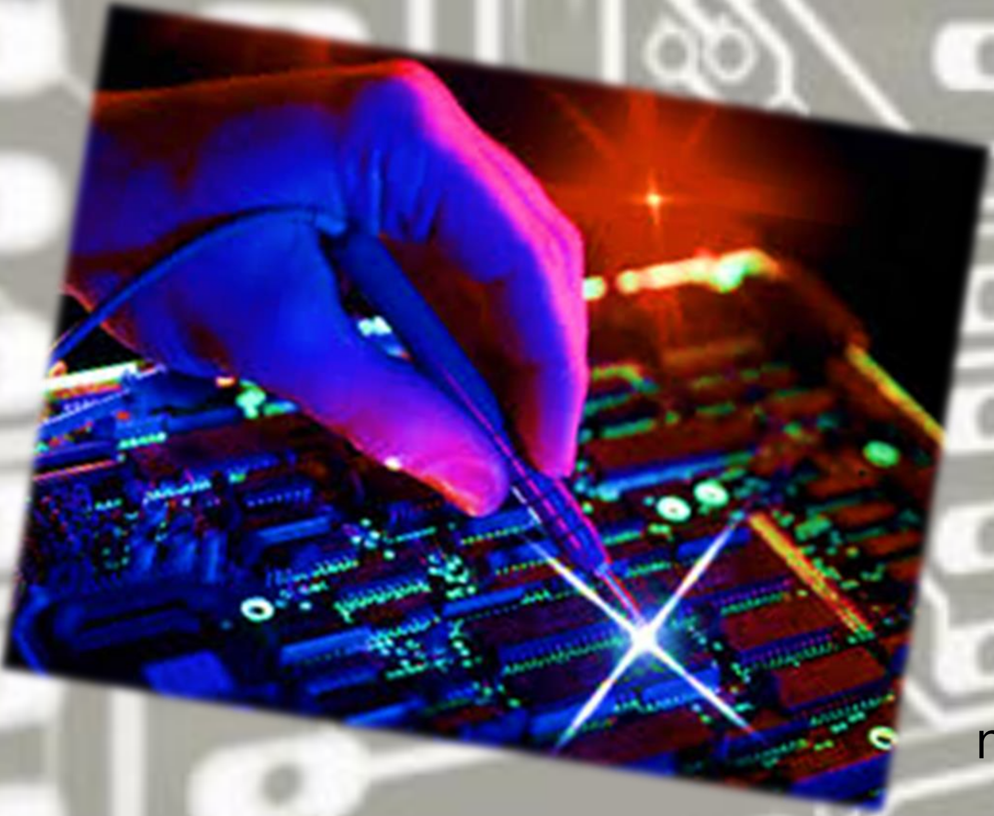
Electric Circuit Theory



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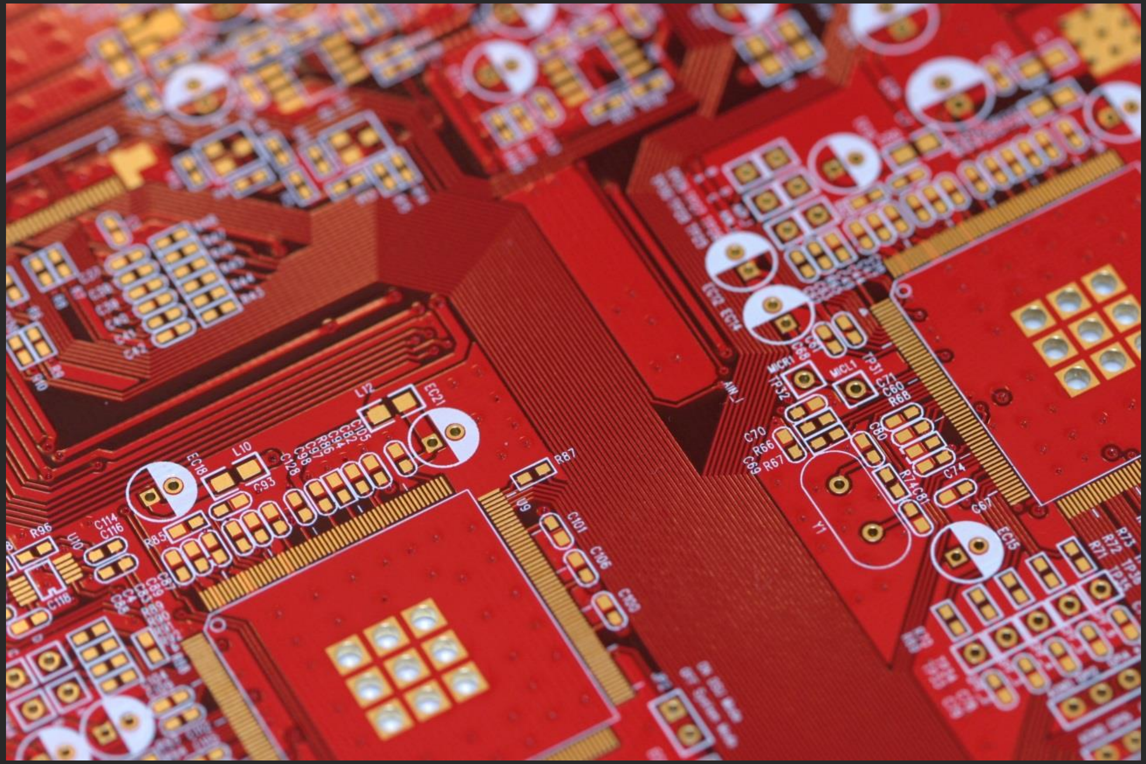
Chapter 8

Natural and Step Responses of *RLC* Circuits



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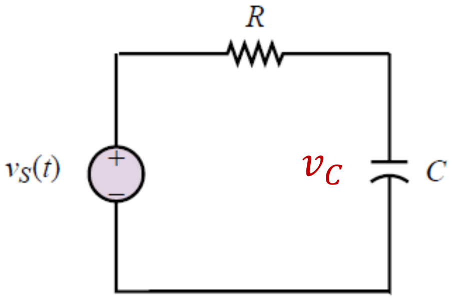
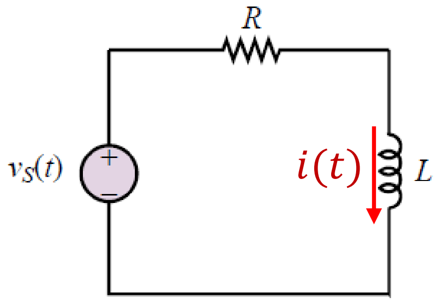
8.1 Introduction to the Natural Response of a Parallel *RLC* Circuit

- **First-Order Circuits (Chapter 7)**
  - Circuits with a single storage element (a capacitor or an inductor).
  - The differential equations describing them are first-order.

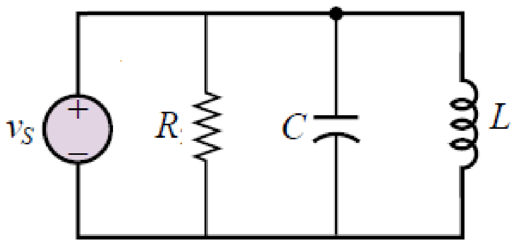


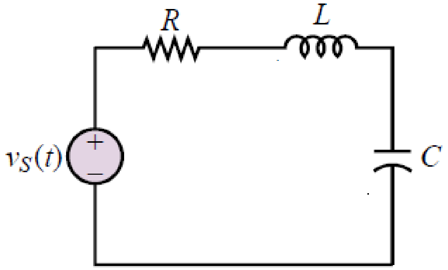


$$\frac{dv_C}{dt} + \frac{v_C - V_s}{RC} = 0 \qquad \frac{di}{dt} + \frac{Ri - V_s}{L} = 0$$

- **Second-Order Circuits (Chapter 8)**
  - Circuits containing two storage elements
  - Their responses are described by differential equations that contain second derivatives.
  - Typical examples of second-order circuits are RLC circuits.

> A second-order circuit is characterized by a second-order differential equation.
>
> It consists of resistors and the equivalent of two energy storage elements.





- There are two basic types of RLC circuits: **parallel connected** and **series connected**.

Finding Initial Values

- There are two key points to keep in mind in determining the initial conditions.
  - We must carefully handle the polarity of voltage $v_C(t)$ across the capacitor and the direction of the current $i_L(t)$ through the inductor.

    Keep in mind that $v_C(t)$ and $i_L(t)$ are defined strictly according to the passive sign convention.

    One should carefully observe how these are defined and apply them accordingly.
  - Keep in mind that the capacitor voltage is always continuous so that

$$v_c(0^+) = v_C(0^-)$$

and the inductor current is always continuous so that

$$i_L(0^+) = i_L(0^-)$$

where $t = 0^-$ denotes the time just before a switching event
and $t = 0^+$ is the time just after the switching event,
assuming that the switching event takes place at t = 0.

- Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly, capacitor voltage and inductor current.

Obtaining of the Differential Equation for a Parallel RLC Circuit

- Parallel RLC circuits find many practical applications, notably in communications networks and filter designs.
- Assume initial inductor current I_0 and initial capacitor voltage V_0,

$$i_L(0) = \frac{1}{L}\int_{-\infty}^{0} v(t)dt = I_o$$

$$v_c(0) = V_o$$

- Since the three elements are in parallel, they have the same voltage v across them. Applying KCL at the top node gives

$$i_R + i_L + i_C = 0$$

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(t)dt + C\frac{dv}{dt} = 0$$

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{0} v(t)dt + \frac{1}{L}\int_{0}^{t} v(t)dt + C\frac{dv}{dt} = 0$$

$$\frac{v}{R} + I_o + \frac{1}{L}\int_{0}^{t} v(t)dt + C\frac{dv}{dt} = 0 \quad (8.1)$$

Obtaining of the Differential Equation for a Parallel RLC Circuit

- Taking the derivative with respect to *t* results in

$$\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = 0 \quad (8.2) \quad \leftarrow \quad \frac{v}{R} + I_o + \frac{1}{L}\int_0^t v(t)dt + C\frac{dv}{dt} = 0 \quad (8.1)$$

- Dividing by C Arranging the derivatives in the descending order, we get

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad (8.3)$$

This is a second-order differential equation and is the reason for calling the RLC circuits in this chapter second-order circuits.

Solution of the Differential Equation

- Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let

$$v(t) = Ae^{st} \quad (8.4)$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad (8.3)$$

where A and s are unknown constants.

- Substituting Eq. (8.4) into Eq. (8.3) and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{Ae^{st}}{LC} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right) = 0 \quad (8.5)$$

Since $e^{st} \neq 0$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad (8.6)$$

We cannot A=0 as a general solution because the voltage is not zero for all time and Ae^{st} is the assumed solution we are trying to find.

Solution of the Differential Equation

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad (8.6)$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad (8.3)$$

- Eq.(8.6) is known as the **characteristic equation** of the differential Eq. (8.3), since the roots of the equation determine the mathematical character of v(t).
- The two roots of Eq. (8.8) are

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad (8.7)$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad (8.8)$$

8.1 Introduction to the Natural Response of a Parallel *RLC* Circuit

- **Solution of the Differential Equation**
  - The two values of s in Eqs.(8.7) and (8.8) indicate that there are two possible solutions for i, each of which is of the form of the assumed solution in Eq. (8.4); that is,

$$v_1 = A_1 e^{s_1 t}$$

$$v_2 = A_2 e^{s_2 t}$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad (8.3)$$

  - Since Eq. (8.3) is a linear equation, any linear combination of the two distinct solutions v_1 and v_2 is also a solution of Eq. (8.3).
  - A complete or total solution of Eq. (8.3) would therefore require a linear combination of v_1 and v_2.
  - Thus, the **natural response** of the parallel RLC circuit is

$$v(t) = v_1 + v_2 = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (8.9)$$

where the constants A_1 and A_2 are determined from the initial values v(0) and dv(0)/dt .

Solution of the Differential Equation

- We can show that Eq.(8.9) also is a solution.

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (8.9)$$

$$\frac{dv}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t} \qquad \frac{d^2v}{dt^2} = A_1 s_1^2 e^{s_1 t} + A_2 s_2^2 e^{s_2 t}$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad (8.3)$$

$$A_1 e^{s_1 t}\left(s_1^2 + \frac{1}{RC}s_1 + \frac{1}{LC}\right) + A_2 e^{s_2 t}\left(s_2^2 + \frac{1}{RC}s_2 + \frac{1}{LC}\right) = 0 \quad (8.12)$$

- Each parenthetical term is zero because by definition s_1 and s_2 are roots of the characteristic equation.

$$\left.\begin{aligned} s_1^2 + \frac{1}{RC}s_1 + \frac{1}{LC} = 0 \\ s_2^2 + \frac{1}{RC}s_2 + \frac{1}{LC} = 0 \end{aligned}\right\} \Longleftrightarrow s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad (8.6)$$

Definition of Frequency Terms

- We have shown that v_1 is a solution, v_2 is a solution, and v_1+v_2 is a solution. Therefore, the general solution of Eq.(8.3) has the form given in Eq.(8.13).

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (8.13), (8.9)$$

- The behavior of v(t) depends on the values of s_1 and s_2.

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad (8.14)$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \qquad (8.15)$$

$$\alpha = \frac{1}{2RC} \qquad (8.16) \qquad\qquad \omega_o = \frac{1}{\sqrt{LC}} \qquad (8.17)$$

- Since the exponents $s_1 t$ and $s_2 t$ must be dimensionless, s_1 and s_2 must have the unit of "per second".

Definition of Frequency Terms

- Therefore, the units of α and ω_o must also be "per second" or s^{-1}. A unit of this type is called **frequency**.
- Let us define new terms:
  - Neper frequency (exponential damping coefficient)

$$\alpha = \frac{1}{2RC} \qquad (8.16)$$

  - Resonant frequency

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad (8.17)$$

  - Complex frequency

$$s_1, s_2$$

Types of Responses(solutions)

- From Eqs. (8.14),(8.15), we can infer that there are three types of solutions depending on the relative sizes of α and ω_o:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \quad (8.14) \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \quad (8.15) \qquad \alpha = \frac{1}{2RC} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

- **Overdamped response :**

  If $\alpha^2 > \omega_o^2$, both roots s_1 and s_2 are negative and real. $L > 4R^2C \rightarrow \alpha > \omega_o$

- **Critically damped response**

  If $\alpha^2 = \omega_o^2$, $s_1 = s_2 = -\alpha$ $\qquad L = 4R^2C \rightarrow \alpha = \omega_o$

- **Underdamped response**

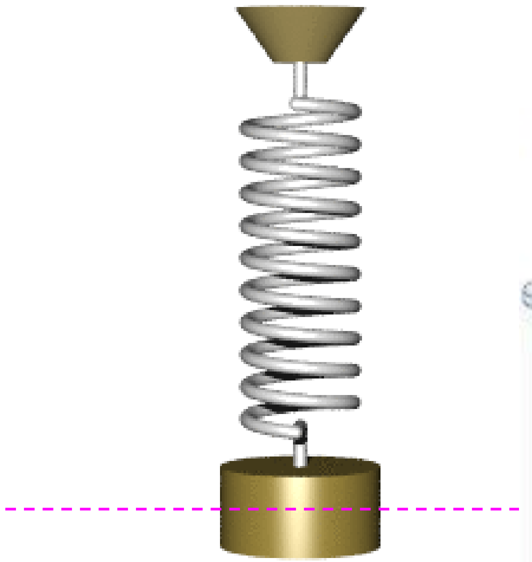
  If $\alpha^2 < \omega_o^2$, both roots s_1 and s_2 are complex and, in addition, are conjugates of each other.

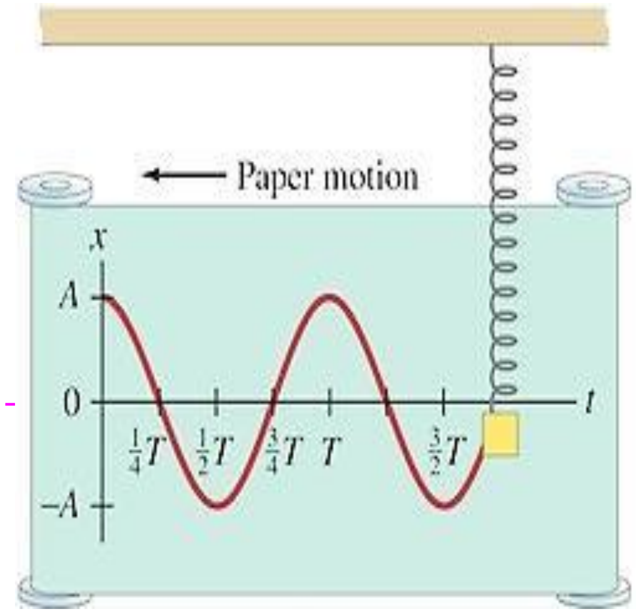
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + \sqrt{-(\omega_o^2 - \alpha^2)} = -\alpha + j\omega_d \qquad L < 4R^2C \rightarrow \alpha < \omega_o$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha - \sqrt{-(\omega_o^2 - \alpha^2)} = -\alpha - j\omega_d$$

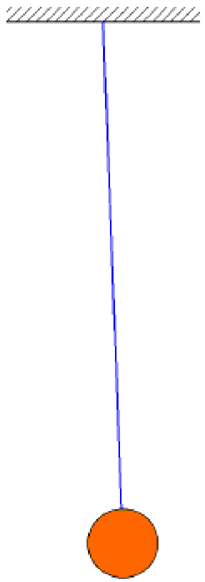
- **What is the damping?**
  - **Undamped System**

    The following physical systems are some examples of **simple harmonic oscillator**.





An undamped spring–mass system undergoes simple harmonic motion



The motion of an undamped pendulum approximates to simple harmonic motion if the angle of oscillation is small

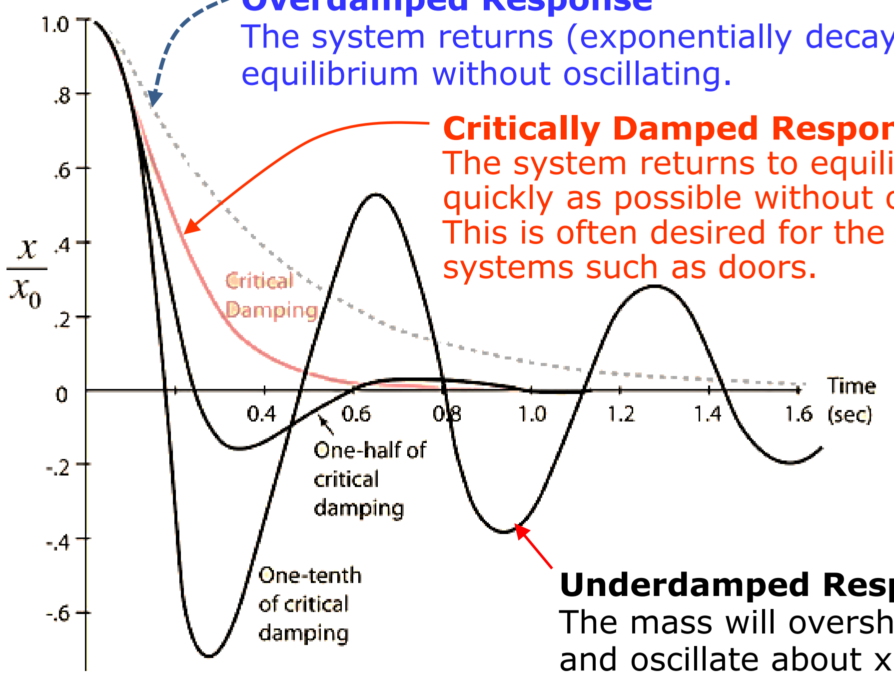
8.2 The Forms of the Natural Response of a Parallel *RLC* Circuit

Damped System

- Damping is a dissipation of energy from a vibrating structure.
- The term dissipate is used to mean the transformation of mechanical energy into other form of energy and, therefore, a removal of mechanical energy from the vibrating system.

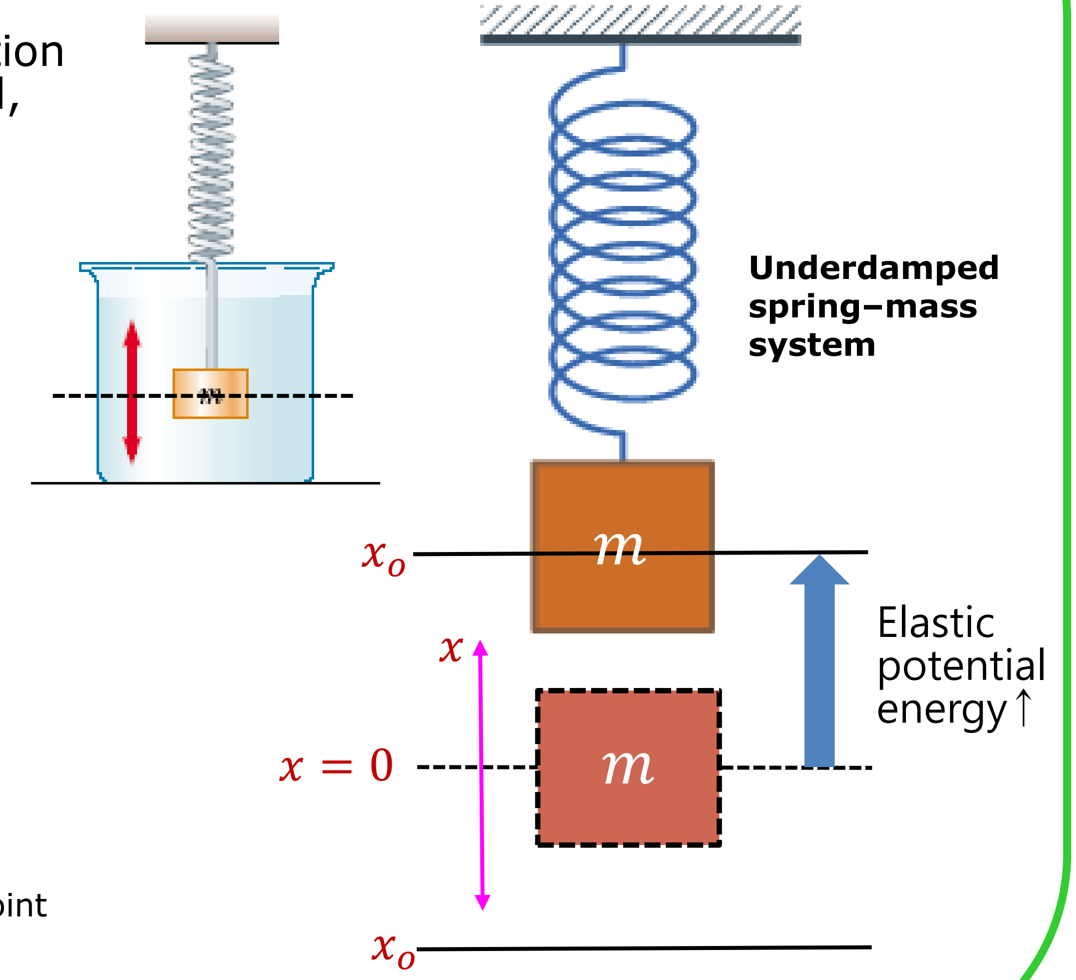
Overdamped Response
The system returns (exponentially decays) to equilibrium without oscillating.

Critically Damped Response
The system returns to equilibrium as quickly as possible without oscillating. This is often desired for the damping of systems such as doors.



Underdamped Response
The mass will overshoot the zero point and oscillate about x=0.

A release from rest at a position x_0.



Damped harmonic oscillator

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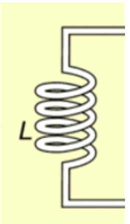
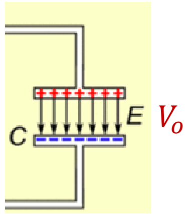
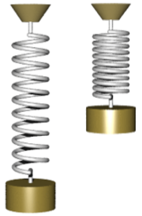
8.2 The Forms of the Natural Response of a Parallel *RLC* Circuit

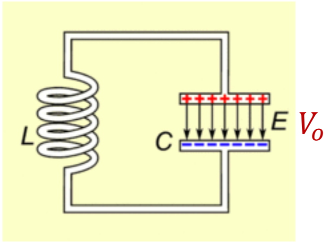
Electrical System

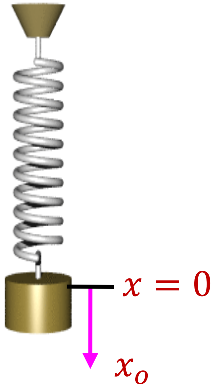
- The capacitor stores energy in its electric field E and the inductor stores energy in its magnetic field B (green).
- The charge flows back and forth between the plates of the capacitor, through the inductor.
- The energy oscillates back and forth between the capacitor and the inductor **until** (if not replenished from an external circuit) **internal resistance makes the oscillations die out**.



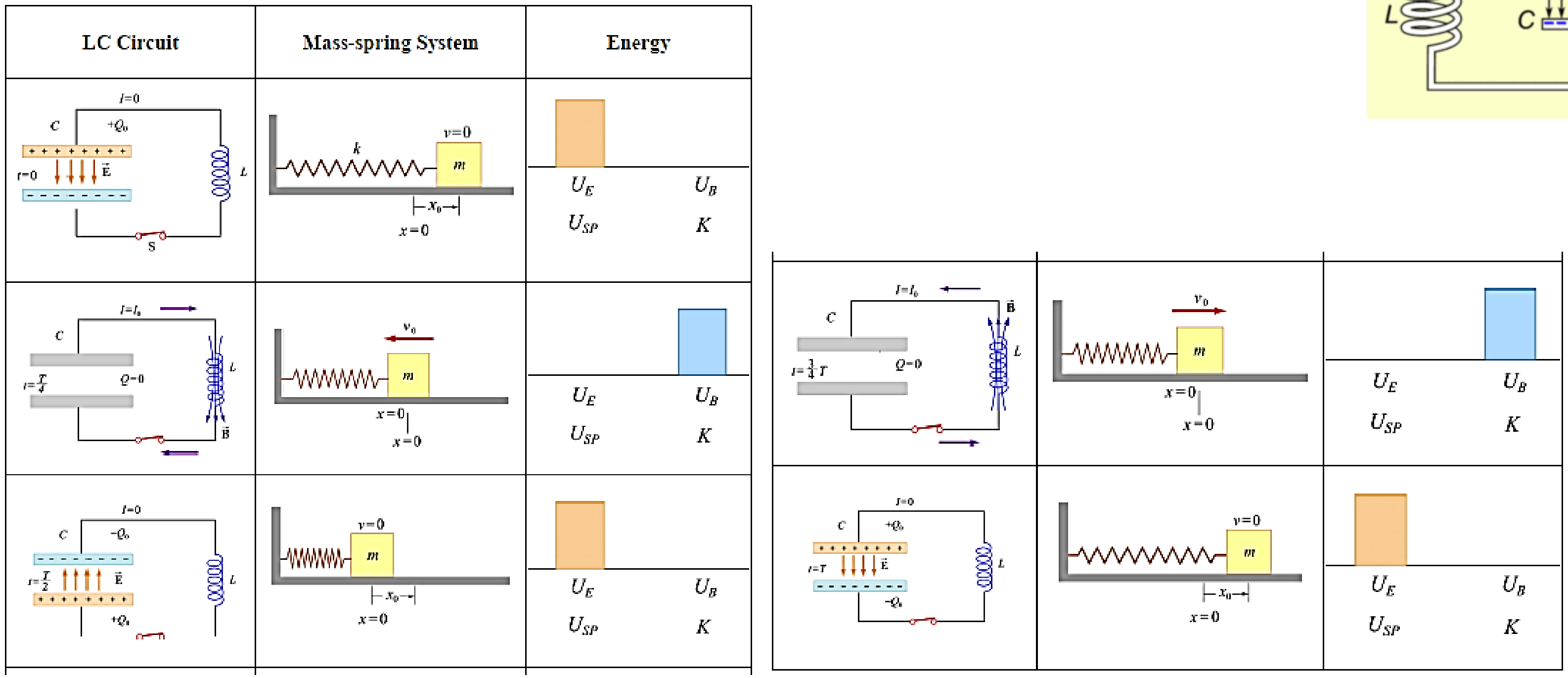






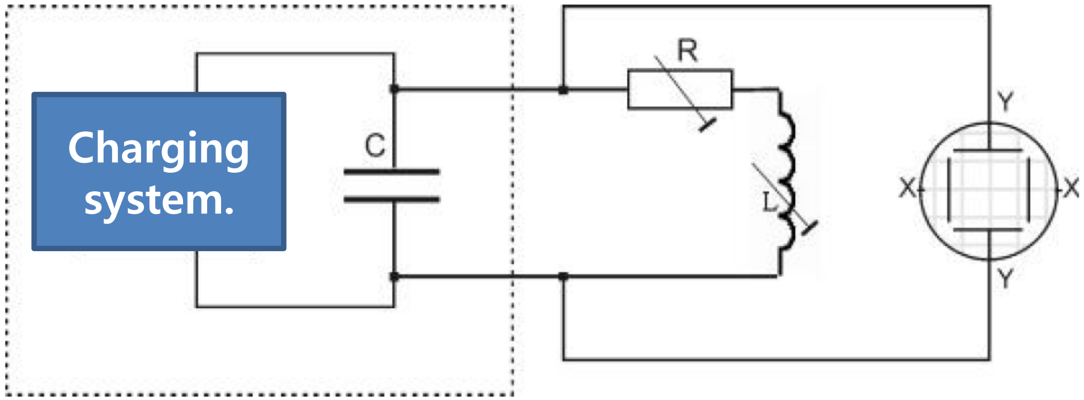


Energy oscillations in the *LC* Circuit and the mass-spring system



8.2 The Forms of the Natural Response of a Parallel *RLC* Circuit

- **Electrical System**
  - We now consider a RLC circuit which contains a resistor, an inductor and a capacitor.
  - Unlike the LC circuit energy will be dissipated through the resistor.



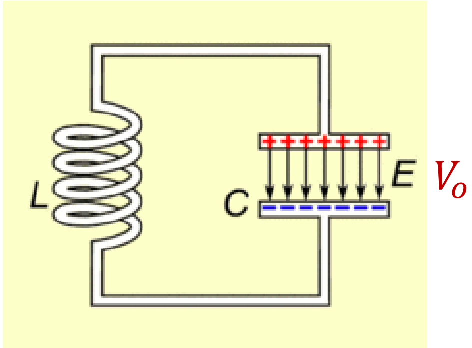
The Overdamped Voltage Response

- If $L > 4R^2C$, $\alpha > \omega_o$.
  The roots of the characteristic equation are negative real numbers.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$\alpha = \frac{1}{2RC} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$



- The response is

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (8.18)$$

- **The process for finding the overdamped response, v(t):**

  (1) Find s_1 and s_2, using the values of R, L, and C.

  (2) Find $v(0^+)$ and $dv(0^+)/dt$.

  (3) Find the values of A_1 and A_2 using $v(0^+)$ and $dv(0^+)/dt$.

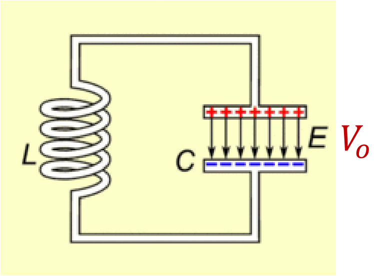
  (4) Substitute the values for s_1 , s_2, A_1 and A_2 into Eq.(8.18).

The Overdamped Voltage Response

The process for finding the overdamped response, v(t):

(2) Find $v(0^+)$ and $dv(0^+)/dt$.

- Two initial conditions



$$v(0^+) = V_o$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} \quad (8.21)$$

$$i_C = C\frac{dv}{dt}$$

$$= -\frac{i_L(0^+) + i_R(0^+)}{C}$$

$$= -\frac{I_o}{C} - \frac{V_o}{CR}$$

$$i_C + i_L + i_R = 0$$

$$i_C(0^+) + i_L(0^+) + i_R(0^+) = 0$$

$$i_C(0^+) = -i_L(0^+) - i_R(0^+) = -I_o - \frac{V_0}{R} \quad (8.22)$$

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- **The Overdamped Voltage Response**
  - **The process for finding the overdamped response, v(t):**

(3) Find the values of A_1 and A_2 using $v(0^+)$ and $dv(0^+)/dt$.

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad (8.18)$$

$$v(0^+) = A_1 + A_2 = V_o \quad (8.23)$$

$$\frac{dv(t)}{dt} = s_1A_1e^{s_1t} + s_2A_2e^{s_2t}$$

$$\frac{dv(0^+)}{dt} = s_1A_1 + s_2A_2 \quad (8.24)$$

8.2 The Forms of the Natural Response of a Parallel *RLC* Circuit

The Underdamped Voltage Response

- If $L < 4R^2C$, $\alpha < \omega_o$.
  The roots of the characteristic equation are complex, and the response is underdamped.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + \sqrt{-(\omega_o^2 - \alpha^2)} = -\alpha + j\sqrt{(\omega_o^2 - \alpha^2)} = -\alpha + j\omega_d \qquad (8.25)$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha - j\omega_d \qquad (8.26)$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} \qquad \alpha = \frac{1}{2RC} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

- The response is

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad (8.28)$$

8.2 The Forms of the Natural Response of a Parallel *RLC* Circuit

The Underdamped Voltage Response

Finding The Underdamped Voltage Response

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad (8.28)$$

The natural response is

$$v(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$$

$$= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

$\leftarrow$ $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, $s_1 = -\alpha + j\omega_d$, $s_1 = -\alpha - j\omega_d$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{-j\theta} = \cos\theta - j\sin\theta \qquad (8.29)$$

We get

$$v(t) = e^{-\alpha t}[A_1(\cos \omega_d t + j \sin \omega_d t) + A_2 (\cos \omega_d t - j \sin \omega_d t)]$$

$$= e^{-\alpha t}[(A_1 + A_2) \cos \omega_d t + j(A_1 - A_2) \sin \omega_d t]$$

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad (8.28)$$

$$B_1 = A_1 + A_2 \qquad B_2 = j(A_1 - A_2)$$

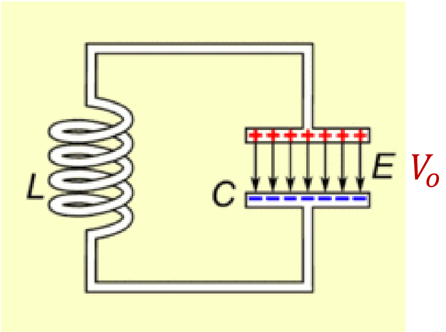
- **The Underdamped Voltage Response**

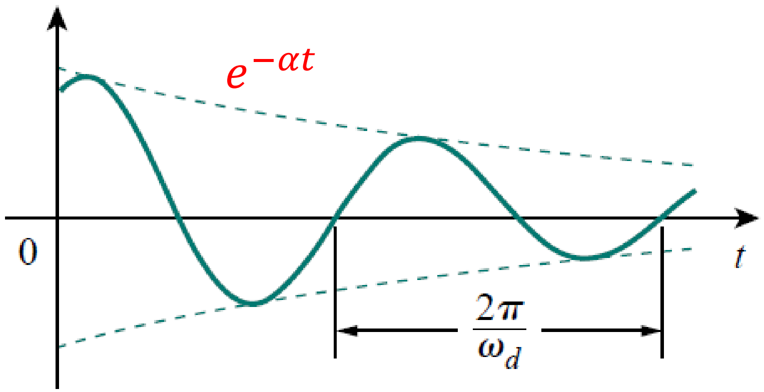
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad (8.28)$$

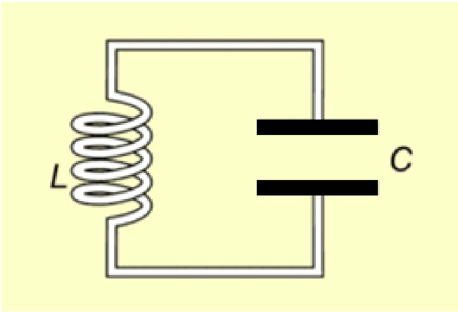
With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature.

The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$.

Figure depicts a typical underdamped response. [Figure assumes for each case that i(0) = 0].







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▪ The Underdamped Voltage Response

➤ Find the values of B_1 and B_2 using $v(0^+)$ and $dv(0^+)/dt$.

$$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \quad (8.28)$$

$$v(0^+) = B_1 + 0 = V_o \quad (8.30)$$

$$\frac{dv(t)}{dt} = (-\alpha e^{-\alpha t})(B_1 \cos\omega_d t) + e^{-\alpha t}(-\omega_d B_1 \sin\omega_d t)$$
$$+(-\alpha e^{-\alpha t})(B_2 \sin\omega_d t) + e^{-\alpha t}(\omega_d B_2 \cos\omega_d t)$$

$$\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 \quad (8.31)$$

The Critically Damped Voltage Response

- If $L = 4R^2C$, $\alpha = \omega_o$.
  The two roots of the characteristic equation are real and equal.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha$$

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC} \qquad (8.32)$$

$$\alpha = \frac{1}{2RC} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

- The response is

$$v(t) = (D_1 t + D_2)e^{-\alpha t} \qquad (8.34)$$

The Critically Damped Voltage Response

Finding the natural response (Eq.8.34)

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \qquad (1) \qquad (8.3)$$

$$\alpha = \omega_o = \frac{1}{2RC} = \frac{1}{\sqrt{LC}}$$

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0 \qquad (2)$$

$$\frac{d}{dt}\left(\frac{dv}{dt} + \alpha v\right) + \alpha\left(\frac{dv}{dt} + \alpha v\right) = 0 \qquad (3)$$

If we let

$$f = \frac{dv}{dt} + \alpha v \qquad (4)$$

then Eq.(3) becomes

$$\frac{df}{dt} + \alpha f = 0 \qquad (5)$$

Eq.(3) is a first-order differential equation with solution

$$f = A_1 e^{-\alpha t} \qquad (6)$$

Eq.(4) then becomes

$$\frac{dv}{dt} + \alpha v = D_1 e^{-\alpha t} \quad \text{or} \quad e^{\alpha t}\frac{dv}{dt} + e^{\alpha t}\alpha v = D_1 \qquad (7)$$

This can be written as

$$\frac{d(e^{\alpha t}v)}{dt} = D_1 \qquad (8)$$

The Critically Damped Voltage Response

Finding the natural response (Eq.8.34)

$$\frac{d(e^{\alpha t}v)}{dt} = D_1 \qquad (8)$$

Integrating both sides yields

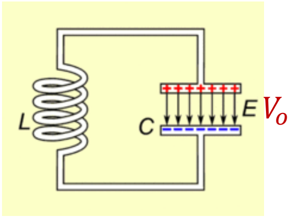
$$d(e^{\alpha t}v) = D_1 dt$$

$$e^{\alpha t}v = D_1 t + D_2 \qquad (9)$$

or

$$v(t) = (D_1 t + D_2)e^{-\alpha t} \qquad (8.34)$$

This is a typical critically damped response.
The natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term.



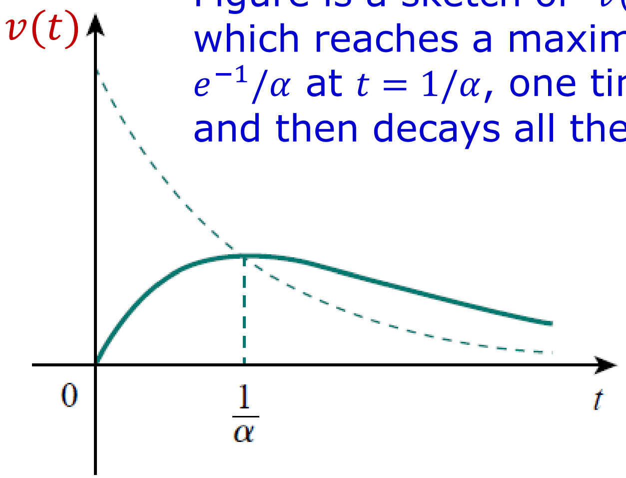


Figure is a sketch of $v(t) = te^{-\alpha t}$, which reaches a maximum value of e^{-1}/α at $t = 1/\alpha$, one time constant, and then decays all the way to zero.

The Critically Damped Voltage Response

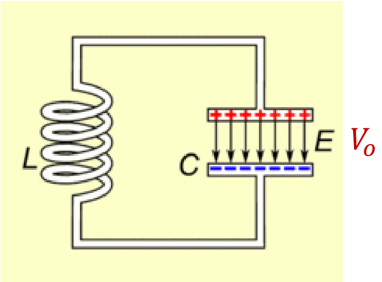
- **Finding the values of D_1 and D_2 using $v(0^+)$ and $dv(0^+)/dt$.**

$$v(t) = (D_1 t + D_2)e^{-\alpha t} \qquad (8.34)$$

$$v(0^+) = 0 + D_2 = V_o \qquad (8.35)$$

$$\frac{dv(t)}{dt} = (D_1 + 0)e^{-\alpha t} + (D_1 t + D_2)(-\alpha e^{-\alpha t})$$

$$\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 \qquad (8.36)$$



Responses of Parallel RLC Circuit

Natural Response: Summary

- The circuit is being excited by the energy initially stored in the capacitor and inductor.
- The energy is represented by the initial capacitor voltage V_o and initial inductor current I_o.
- Thus, at t = 0,

$$v_C(0) = \frac{1}{C}\int_{-\infty}^{0} i(t)dt = V_o \qquad i(0) = I_o$$

- **Overdamped response:** $\alpha^2 > \omega_o^2 \quad L > 4R^2C$

  Both roots s_1 and s_2 are negative and real.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \qquad v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- **Underdamped response:** $\alpha^2 < \omega_o^2 \quad L < 4R^2C$

  The roots of the characteristic equation are complex.

$$s_1 = -\alpha + j\omega_d \qquad s_2 = -\alpha - j\omega_d \qquad v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

- **Critically damped response:** $\alpha^2 = \omega_o^2 \quad L = 4R^2C$

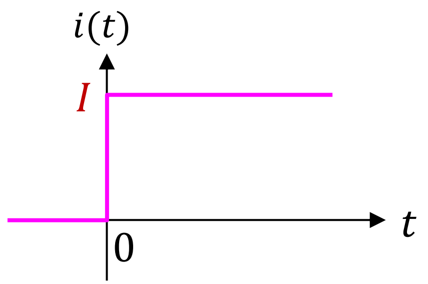
  The two roots of the characteristic equation are real and equal.

$$s_1 = s_2 = -\alpha \qquad v(t) = (D_1 t + D_2)e^{-\alpha t}$$

Responses of Parallel RLC Circuit

Step Response

- The step response is obtained by the sudden application of a dc source.



- We want to find i due to a sudden application of a dc current.

- To develop a general approach to finding the step response of a second order circuit, we focus on finding the current in the inductor branch, i_L.
  This current does not approach zero as t increases.

$$i_L(\infty) = I$$

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Step Response

- Applying KCL at the top node for t > 0,

$$i_L + i_R + i_C = I$$

or

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I \qquad (8.37)$$

- From the definition

$$v = L\frac{di_L}{dt} \qquad (8.38)$$

$$\frac{dv}{dt} = L\frac{d^2 i_L}{dt^2} \qquad (8.39)$$

- Substituting Eqs.(8.38) and (8.39) into Eq.(8.37) gives

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I \qquad (8.40)$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC} \qquad (8.41)$$

Step Response

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC} \qquad (8.41)$$

- The complete solution to Eq. (8.41) consists of the natural response i_n and the forced response i_f; that is,,

$$i_L(t) = i_n(t) + i_f(t)$$

- The natural response is the same as what we had in Section 8.2.

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ | $i_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$ | $i_n(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$

$v(t) = (D_1 t + D_2)e^{-\alpha t}$ | $i_n(t) = (D_1 t + D_2)e^{-\alpha t}$

- The forced response is the steady state or final value of i.
  In the circuit in Fig.8.11, the final value of the current through the inductor is the same as the source current I. Thus,

$$i_f(t) = I$$

Step Response

- The complete solutions:

$$i_L(t) = i_n(t) + i_f(t)$$

- **Overdamped response**

$$\left.\begin{aligned} i_n(t) &= A_1 e^{s_1 t} + A_2 e^{s_2 t} \\ i_f(t) &= I \end{aligned}\right\} \quad i_L(t) = I + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (8.47)$$

- **Underdamped response**

$$\left.\begin{aligned} i_n(t) &= e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \\ i_f(t) &= I \end{aligned}\right\} \quad i_L(t) = I + e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad (8.48)$$

- **Critically damped response**

$$\left.\begin{aligned} i_n(t) &= (D_1 t + D_2) e^{-\alpha t} \\ i_f(t) &= I \end{aligned}\right\} \quad i_L(t) = I + (D_1 t + D_2) e^{-\alpha t} \qquad (8.49)$$