Time Series Analysis

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Spectral Analysis

- We view a time series as being composed of periodic components.
- Methods we can use
  - Fourier transform: frequency domain analysis
    suitable without apparent sharp-discontinuity
  - Wavelet transform: time-frequency domain analysis
    working well even with sharp-discontinuity
- What can we do?
  - Frequency detection
  - Noise removal
  - Model detection
  - Lag detection
  - ...

Spectral density

- If $\gamma_x(h)$ of a stationary process satisfies $\sum_{h=-\infty}^{\infty} |\gamma_x(h)| < \infty$, then spectral density $f(w)$ is given by

$$f(w) = \sum_{h=-\infty}^{\infty} \gamma_x(h) e^{-2\pi i w h},$$

where $-1/2 \le w \le 1/2$. [Property 4.2]

- Then the inverse transform of the spectral density is given by

$$\gamma_x(h) = \int_{-1/2}^{1/2} e^{2\pi i w h} f(w) dw,$$

where $h = 0, \pm 1, \pm 2, \cdots$.

- Notice that $\gamma_x(h)$ on time domain and $f(w)$ on frequency domain are interchangeable.

Spectral density (cont.)

- The following are useful:

$$
\begin{aligned}
|a+bi| &= \sqrt{a^2+b^2},\\
e^{\theta i} &= \cos\theta + i\sin\theta,\\
\sin\theta = -\sin -\theta &= \frac{e^{\theta i}+e^{-\theta i}}{2}\\
\cos\theta = \cos -\theta &= \frac{e^{\theta i}-e^{-\theta i}}{2i}
\end{aligned}
$$

- The proof of the inverse:

$$
\begin{aligned}
(R.H.S.) &= \int_{-1/2}^{1/2} e^{2\pi iwh} f(w)dw = \int_{-1/2}^{1/2} e^{2\pi iwh} \sum_{p=-\infty}^{\infty} \gamma_x(p)e^{-2\pi iwp}dw\\
&= \sum_{p=-\infty}^{\infty} \gamma_x(p) \int_{-1/2}^{1/2} e^{2\pi iw(h-p)} = \gamma_x(h).
\end{aligned}
$$

Spectral density (cont.)

- Also we have $\gamma(0) = Var(X_t) = \int_{-1/2}^{1/2} f(w)dw$.
- Simply speaking, the information of $\gamma_x \equiv$ the information of $f_x(w)$.
- Since $f(w) = f(-w)$, we can focus on $0 \leq w \leq 1/2$.
- [e.g.] White noise $WN(0, \sigma_a^2)$ has

$$f_a(w) = \sigma_a^2$$

  because $\gamma_a(h) = \sigma_a^2$ only when $h = 0$.
- [e.g.] ARMA(p, q) $\phi(B)x_t = \theta(B)a_t$, where $\phi(B)$ is a polynomial of order p and $\theta(B)$ of order q, has

$$f_x(w) = \sigma_a^2 \frac{|\theta(e^{-2\pi wi})|^2}{|\phi(e^{-2\pi wi})|^2}.$$

Spectral density (cont.)

- [Example 4.3] For a periodic stationary random process

$$x_t = U_1 \cos(2\pi w_0 t) + U_2 \sin(2\pi w_0 t),$$

where U_1 and U_2 are independent with zero mean and σ^2 variance, spectral distribution $F_x(w)$ is given by a step function with jumps at $-w$ and w.
So spectral density $f(w)$, which is the differentiation of $F_x(w)$, has two spots of point mass at $-w$ and w.

- [e.g.] MA(1) $x_t = a_t + 0.5a_{t-1}$.

$$\gamma_x(h) = \begin{cases} (1+0.5^2)\sigma_a^2, & h = 0 \\ 0.5\sigma_a^2, & h = \pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_x(w) = \sum_{h=-\infty}^{\infty} \gamma_x(h) e^{-2\pi i w h} = \sigma_a^2[1.25 + 0.5(e^{-2\pi i w} + e^{2\pi i w})]$$
$$= \sigma_a^2[1.25 + \cos 2\pi w].$$

Spectral density (cont.)

- Or we can directly use the previous result of ARMA(p, q).
- [e.g.] AR(2) $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + a_t, \phi_1 = 1, \phi_2 = -0.9$.

$$
\begin{aligned}
f_x(w) &= \sigma_a^2 \frac{1}{|1 - \phi_1 e^{-2\phi i w} - \phi_2 (e^{-2\phi i w})^2|^2} \\
&= \sigma_a^2 \frac{1}{2.81 - 3.8\cos(2\phi w) + 1.8\cos(4\phi w)}.
\end{aligned}
$$

Spectral density (cont.)

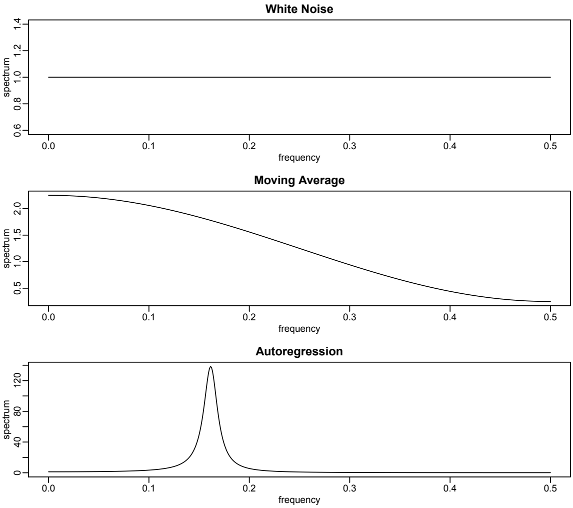

Figure : Theoretical spectral densities