제 3 강
2계 상미분방정식

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1.2계 선형 상미분방정식(Linear D. E. of second order)

$$y'' + p(x)y' + q(x)y = r(x)$$

1) $r(x) = 0$: 2계 선형 제차 상미분방정식

2) $r(x) \neq 0$: 2계 선형 비제차 상미분방정식

2계 선형 제차 상미분방정식

$$y'' + p(x)y' + q(x)y = 0$$

i) 일반해 : $y_h = c_1 y_1 + c_2 y_2$ (y_1, y_2 : 기본근 (basis), c_1, c_2 : 상수(constant))

ii) 특별해 : y_p (초기값 조건 $y(0) = a,\ y'(0) = b$, 경계값 조건 $y(a) = A,\ y(b) = B$)

2계 선형 비제차 상미분방정식

$$y'' + p(x)y' + q(x)y = r(x)$$

i) 일반해 : $y = y_h + y_p$ (y_p : $r(x)$에 대한 근)

$$= c_1 y_1 + c_2 y_2 + y_p$$

(주의 : y_1, y_2, y_p는 비제차 미방의 기본근이다.

따라서 비례해서는 안된다.)

ii) 특별해 : y_h의 특별해와 같음

계수 축소법 (method of reduction of order)

$$y'' + p(x)y' + q(x)y = 0 \ : \ \frac{y_1}{y_2} \neq k \, (Not \ \ proportional)$$

$$y_1 \, (known) \qquad \rightarrow \qquad y_2 \, (unknown)$$

$$\frac{y_2}{y_1} = u(x). \qquad\qquad y_2 = u \cdot y_1$$

$$y_2' = u' y_1 + u y_1', \qquad y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$$

$$= u'' y_1 + 2u' y_1' + u y_1''$$

2. 다른 근 구하기 (How to obtain another basis if one basis is known)

$$y''+py'+qy=u''y_1+2u'y_1'+uy_1''+p(u'y_1+uy_1')+quy_1$$
$$=y_1\cdot u''+(2y_1'+py_1)u'+(y_1''+py_1'+qy_1)u=0$$

$$y_1\cdot u''+(2y_1'+py_1)u'=0 \quad , \quad u'=U, \quad u''=U'$$

$$y_1\cdot U'+(2y_1'+py_1)U=0 \leftarrow \textit{Separable form}.$$

$$\int\frac{dU}{U}=\int-\frac{2y_1'+py_1}{y_1}dx$$

$$\ln|U|=-\int\left(\frac{2y_1'}{y_1}+p\right)dx\ ,\quad U=\exp\left(-\int\left(\frac{2y_1'}{y_1}+p\right)dx\right)$$

$$u'=U=\exp\left(-\int\left(\frac{2y_1'}{y_1}+p\right)dx\right)$$

$$\therefore u=\int U\,dx\ ,\quad y_2=\int U\,dx\cdot y_1$$

3. 2계 상수계수 선형 제차 상미분방정식(Second order homogeneous ordinary D.E. with constant coefficients)

$$y'' + ay' + by = 0$$

해법 : $y' + cy = 0$, $y = e^{-cx} = e^{\lambda x}$ (-c = λ로 치환)

$$y = e^{\lambda x},\ y' = \lambda e^{\lambda x},\ y'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = (\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

$\lambda^2 + a\lambda + b = 0$: λ의 특성 2차 방정식

3. 2계 상수계수 선형 제차 상미분방정식(Second order homogeneous ordinary D.E. with constant coefficients)

$$\lambda^2 + a\lambda + b = 0, \qquad \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$i)\ a^2 - 4b > 0,$

$(distinct\ \ real\ \ roots;\ \lambda_1, \lambda_2)$

$: \quad y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x}$

$\therefore \quad y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$ii)\ a^2 - 4b = 0,$

$(real\ double\ \ root;\ \lambda_1 = \lambda_2 = \lambda)$

$: y_1 = e^{\lambda x},\ y_2 = u y_1 = x \cdot e^{\lambda x} \quad (u(x) = x)$

$\therefore \quad y_h = (c_1 + c_2 x) e^{\lambda x}$

$iii)\ a^2 - 4b < 0\ ,$

$(Complex\ \ conjugate\ \ roots;\ \lambda_{1,2} = \alpha \pm i\beta)$

$$y_1 = e^{\lambda_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x} \cdot e^{i\beta x} = e^{\alpha x} \cdot (\cos \beta x + i \sin \beta x)$$

$$y_1 = e^{\alpha x} \cdot \cos \beta x$$

$$y_2 = e^{\lambda_2 x} = e^{(\alpha - i\beta)x} = e^{\alpha x} \cdot e^{i(-\beta x)} = e^{\alpha x} \cdot (\cos \beta x - i \sin \beta x)$$

$$y_2 = e^{\alpha x} \cdot \sin \beta x$$

$$y_h = (c_1 \cdot \cos \beta x + c_2 \cdot \sin \beta x) e^{\alpha x}$$

예제 1 $16y'' - \pi^2 y = 0.$

sol

$$y'' - \frac{\pi^2}{16} y = 0$$

$$\lambda^2 - \frac{\pi^2}{16} = (\lambda + \frac{\pi}{4})(\lambda - \frac{\pi}{4}) = 0$$

$$\lambda_1 = -\frac{\pi}{4}, \quad \lambda_2 = \frac{\pi}{4}$$

$$y_1 = e^{-\frac{\pi}{4}x}, \quad y_2 = e^{\frac{\pi}{4}x}$$

$$y_h = c_1 e^{-\frac{\pi}{4}x} + c_2 e^{\frac{\pi}{4}x}$$

- 예제 2 $y'' + 2ky' + k^2 y = 0.$

sol

$$\lambda^2 + 2k\lambda + k^2 = (\lambda + k)^2 = 0$$

$$\lambda = -k \text{ , 중근}$$

$$y_1 = e^{-kx}, \quad y_2 = x \cdot e^{-kx}$$

$$y_h = c_1 e^{-kx} + c_2 x \cdot e^{-kx}$$

$$= (c_1 + c_2 x) e^{-kx}$$

- 예제 3 $y'' + 4y' + (4+\omega^2)y = 0$

sol

$$\lambda^2 + 4\lambda + (4+\omega^2) = 0$$

$$\lambda = -2 \pm \sqrt{4-(4+\omega^2)} = -2 \pm \sqrt{-\omega^2}$$

$$= -2 \pm i\omega$$

$$\lambda_1 = -2 + i\omega, \qquad \lambda_2 = -2 - i\omega$$

$$y_1 = e^{-2x} \cdot \cos \omega x, \quad y_2 = e^{-2x} \cdot \sin \omega x$$

$$\therefore\ y_h = (c_1 \cos \omega x + c_2 \sin \omega x)e^{-2x}$$

3. 2계 상수계수 선형 제차 상미분방정식(Second order homogeneous ordinary D.E. with constant coefficients)

- 예제 4 $y'' - 2y' + (4\pi^2 + 1)y = 0, \quad y(0) = -2, \quad y'(0) = 6\pi - 2$

sol

$$\lambda^2 - 2\lambda + (4\pi^2 + 1) = 0$$

$$\lambda = 1 \pm \sqrt{1 - (4\pi^2 + 1)} = 1 \pm \sqrt{-4\pi^2} = 1 \pm i2\pi$$

$$\begin{pmatrix} \lambda_1 = 1 + i \cdot 2\pi, & \lambda_2 = 1 - i \cdot 2\pi \\ y_1 = e^x \cdot \cos 2\pi x, & y_2 = e^x \cdot \sin 2\pi x \end{pmatrix}$$

$$y_h = (c_1 \cos 2\pi x + c_2 \sin 2\pi x)e^x$$

초기값 조건

$$y(0) = -2 = c_1$$

$$y'(0) = 6\pi - 2 = ((-2\cos 2\pi x + c_2 \sin 2\pi x)e^x)'_{x=0}$$

$$= ((4\pi \sin 2\pi x + 2\pi c_2 \cos 2\pi x)e^x + (-2\cos 2\pi x + c_2 \sin 2\pi x)e^x)_{x=0}$$

$$= 2\pi c_2 - 2 \qquad \therefore c_2 = 3$$

$$\therefore y_p = (-2\cos 2\pi x + 3\sin 2\pi x)e^x$$

4. 미분연산자(Differential operator)

$$D = \frac{d}{dx}\ , \quad Dy = \frac{dy}{dx} = y', \quad D_y^2 = D(D_y) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''$$

$$L = P(D) = D^2 + aD + b$$

$$\begin{aligned} L(y) &= P(D)(y) \\ &= (D^2 + aD + b)y \\ &= D^2y + aDy + by \\ &= y'' + ay' + by \end{aligned}$$

- 예제 5 $(64D^2 + 16D + 1)y = 0$

sol

$$64y'' + 16y' + y = 0$$

$$y'' + \frac{1}{4}y' + \frac{1}{64}y = 0$$

$$\lambda^2 + \frac{1}{4}\lambda + \frac{1}{64} = (\lambda + \frac{1}{8})^2 = 0$$

$$\lambda = -\frac{1}{8},$$

$$y_1 = e^{-\frac{1}{8}x}, \quad y_2 = x \cdot e^{-\frac{1}{8}x}$$

$$y_h = (c_1 + c_2 \cdot x) \cdot e^{-\frac{1}{8}x}$$

5. 2계 선형 제차 상미분방정식 / Euler-Cauchy equation

$$x^2 y'' + axy' + by = 0$$

$$y = x^m, \quad y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}$$

$$x^2 \cdot m(m-1)x^{m-2} + ax \cdot mx^{m-1} + bx^m$$

$$= m(m-1)x^m + amx^m + bx^m$$

$$= \{m(m-1) + am + b\}x^m = 0$$

$$x^m \neq 0, \quad m(m-1) + am + b = 0$$

$$m^2 + (a-1)m + b = 0$$: m의 특성 방정식

5. 2계 선형 제차 상미분방정식 / Euler-Cauchy equation

$m^2 + (a-1)m + b = 0$ $\quad m_{1,2} = \dfrac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$

$i)\quad (a-1)^2 - 4b > 0,$

$(distinct\ \ real\ \ roots\ \ ;\ m_1,\ m_2)$

$y_1 = x^{m_1},\quad y_2 = x^{m_2}$

$y_h = c_1 x^{m_1} + c_2 x^{m_2}$

$ii)\quad (a-1)^2 - 4b = 0,$

$(real\ \ double\ \ root\ \ ;\ m_1 = m_2 = m)$

$\dfrac{y_2}{y_1} \neq k,\ y_2 = u(x) \cdot y_1 \quad (u(x) = \ln|x|)$

$y_1 = x^m,\quad y_2 = x^m \cdot \ln|x|$

$y_h = (c_1 + c_2 \cdot \ln|x|)x^m$

- 예제 6 $10x^2y'' + 46xy' + 32.4y = 0$

sol

$$10m(m-1) + 46m + 32.4$$

$$: 10m^2 + 36m + 32.4 = 0$$

$$m = \frac{-18 \pm \sqrt{18^2 - 324}}{10} = \frac{-18 \pm 0}{10}$$

$$= -\frac{9}{5}, \text{ 중근}$$

$$y_1 = x^{-\frac{9}{5}}, \quad y_2 = x^{-\frac{9}{5}} \cdot \ln|x|, \quad (u(x) = \ln|x|)$$

$$y_h = c_1 x^{-\frac{9}{5}} + c_2 x^{-\frac{9}{5}} \cdot \ln|x|$$

- 예제 7 $(xD^2 + D)y = 0$

sol

$xy'' + y' = 0$ (양변에 x를 곱하여)

$x^2y'' + xy' = 0$: $Euler-Cauchy\ \ equation$

$m(m-1) + m = m^2 = 0\,, \quad m = 0\,,$

$y_1 = x^0 = 1\,, \quad y_2 = \ln|x|$

$y_h = c_1 + c_2 \ln|x|$

- 미정 계수법(method of undetermined coefficients)

$$y'' + p(x)y' + q(x)y = r(x)$$

일반해 : $y = y_h + y_p$ (여기서 y_p는 $r(x)$의 근)

$$y'' + ay' + by = r(x)$$

주의 : y_h를 구한 후, y_p를 구한다.

이는 y_1, y_2, y_p가 기본근으로 존재하여야 함으로 비례해서는 안되기 때문이다.

■ 미정 계수법(method of undetermined coefficients)

$Term\ in\ r(x)$	$Choice\ for\ y_p$
ke^{rx}	ce^{rx}
$kx^n\ (n=0, 1, 2, 3\cdots)$	$k_n x^n + k_{n-1}x^{n-1} + \cdots + k_1 x^1 + k_0$
$k\cos wx$	$K\cos wx + M\sin wx$
$k\sin wx$	$K\cos wx + M\sin wx$
$ke^{rx}\cdot\cos wx$	$(K\cos wx + M\sin wx)e^{rx}$
$ke^{rx}\cdot\sin wx$	$(K\cos wx + M\sin wx)e^{rx}$

- 예제 8 $y'' - y = 2e^{x} + 6e^{2x}$

sol

$$\lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0$$

$\lambda_{1,2} = -1,\ 1$ 두 실근

$$y_1 = e^{-x},\quad y_2 = e^{x}$$

$y_p = k_1 x e^{x} + k_2 e^{2x}$ (e^{x}가 중근)

$$y'_p = k_1 e^{x} + k_1 x e^{x} + 2k_2 e^{2x}$$

$$y''_p = k_1 e^{x} + k_1 e^{x} + k_1 x e^{x} + 4k_2 e^{2x}$$

$$2k_1 e^{x} + k_1 x e^{x} + 4k_2 e^{2x} - k_1 x e^{x} - k_2 e^{2x}$$

$$= 2k_1 e^{x} + 3k_2 e^{2x} = 2e^{x} + 6e^{2x}$$

$$2k_1 = 2,\quad k_1 = 1$$

$$3k_2 = 6,\quad k_2 = 2$$

$$y_p = x e^{x} + 2e^{2x}$$

$$y = c_1 e^{-x} + c_2 e^{x} + x e^{x} + 2e^{2x}$$

- 변수변환법(Method of variation of parameters)

$$y'' + p(x)y' + q(x)y = r(x)$$

$r(x)$에 대한 근, y_p를 구하는 가장 일반적 해법이다.

$$y_p(x) = -y_1 \int \frac{y_2 \cdot r}{W} dx + y_2 \int \frac{y_1 \cdot r}{W} dx$$

$$\underset{\text{(Wronskian)}}{W} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_2 \cdot y_1'$$

or

$$y_p(x) = y_1 \int \frac{W_1 \cdot r}{W} dx + y_2 \int \frac{W_2 \cdot r}{W} dx$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

■ 변수변환법(Method of variation of parameters)

$y'' + py' + qy = r(x)$의 y_p에 대한 증명

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$
$$y'_p = u'y_1 + uy_1' + v'y_2 + vy_2' \quad (u'y_1 + v'y_2 = 0)$$
$$= uy_1' + vy_2'$$
$$y''_p = u'y_1' + uy_1'' + v'y_2' + vy_2''$$

$$u'y_1' + uy_1'' + v'y_2' + vy_2'' + puy_1' + pvy_2' + quy_1 + qvy_2$$
$$= (y_1'' + py_1' + qy_1)u + (y_2'' + py_2' + qy_2)v + u'y_1' + v'y_2'$$
$$= r(x)$$

$$\begin{cases} u'y_1' + v'y_2' = r \\ u'y_1 + v'y_2 = 0 \end{cases}$$

$$u'(y_1'y_2 - y_1y_2') = r \cdot y_2 \,. \quad W = y_1y_2' - y_1'y_2$$

$$u' = -\frac{r \cdot y_2}{W}, \quad u = -\int \frac{y_2 \cdot r}{W}dx, \quad v = \int \frac{y_1 \cdot r}{W}dx$$

$$\therefore\ y_p = -y_1 \cdot \int \frac{y_2 \cdot r}{W}dx + y_2 \int \frac{y_1 \cdot r}{W}dx$$

■ 예제 9 $y'' - 4y' + 5y = e^{2x} \cdot \operatorname{cosec} x$

sol

$$\lambda^2 - 4\lambda + 5 = 0$$

$\lambda = 2 \pm \sqrt{4-5} = 2 \pm i$, 두 공액복소근

$$y_1 = e^{2x} \cdot \cos x, \quad y_2 = e^{2x} \cdot \sin x$$

$$W = \begin{vmatrix} e^{2x} \cdot \cos x & , & e^{2x} \cdot \sin x \\ 2e^{2x} \cdot \cos x - e^{2x} \cdot \sin x & , & 2e^{2x} \cdot \sin x + e^{2x} \cdot \cos x \end{vmatrix}$$

$$= e^{2x} \cdot \cos x\,(2e^{2x} \cdot \sin x + e^{2x} \cdot \cos x) - e^{2x} \cdot \sin x\,(2e^{2x} \cdot \cos x - e^{2x} \cdot \sin x)$$

$$= e^{4x}(\cos^2 x + \sin^2 x) = e^{4x}$$

$$y_p = -e^{2x} \cdot \cos x \int \frac{e^{2x} \cdot \sin x \cdot e^{2x} \cdot \dfrac{1}{\sin x}}{e^{4x}} dx + e^{2x} \cdot \sin x \int \frac{e^{2x} \cdot \cos x \cdot e^{2x} \cdot \dfrac{1}{\sin x}}{e^{4x}} dx$$

$$= -e^{2x} \cdot x \cdot \cos x + e^{2x} \cdot \sin x \int \cot x\, dx$$

$$= -e^{2x} \cdot x \cdot \cos x + e^{2x} \cdot \sin x \cdot \ln|\sin x|$$

$$y = (c_1 \cos x + c_2 \sin x)e^{2x} - e^{2x} \cdot x \cdot \cos x + e^{2x} \cdot \sin x \cdot \ln|\sin x|$$